ONE BIT TECHNIQUES

Consider a Gaussian distributed random variable of zero mean, $\underline{X}$; i.e.

$$\text{Prob}\{x < X < x+dx\} = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{x^2}{2\sigma^2}}\, dx = f(x)\,dx.$$

Two random variables, X and Y, are said to possess multinormal statistics if their joint probability function is an elliptical Gaussian distribution,

$$\text{Prob}\{[x < X < x+dx] \text{ and } [y < Y < y+dy]\} = f(x,y)\,dx\,dy$$

$$f(x,y) = A \exp - \left[(x^2 - 2\rho xy + cy^2)(2a^2)^{-1}\right]$$

If we also require symmetry, i.e. X and Y are similarly distributed random variables, c=1.

We may integrate the joint probability function to find the distribution of one variable

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} f(x,y)\,dy \\ &= A \exp - \left[x^2(1-\rho^2)(2a^2)^{-1}\right] \int_{-\infty}^{\infty} \exp - \left[(y-\rho x)^2 (2a^2)^{-1}\right] dy \\ &= A\sqrt{2\pi a^2}\ \exp - \left[x^2(1-\rho^2)(2a^2)^{-1}\right] \end{aligned}$$

Equating this with the normal distribution function of one variable, we have

$$\begin{aligned} a^2 &= \sigma^2(1-\rho^2) \\ A &= \frac{1}{2\pi\sigma^2}(1-\rho^2)^{-\frac{1}{2}} \\ f(x,y) &= \frac{1}{2\pi\sigma^2}(1-\rho^2)^{-\frac{1}{2}} \exp - \left[(x^2 - 2\rho xy + y^2)(2\sigma^2(1-\rho^2))^{-1}\right] \end{aligned}$$

It is of interest to find

$$\begin{aligned} \langle XY \rangle &= \iint_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy \\ &= \frac{1}{2\pi\sigma^2}(1-\rho^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} x \exp - [x^2/2\sigma^2] \int_{-\infty}^{\infty} y \exp - \left[(y-\rho x)^2 (2\sigma^2(1-\rho^2))^{-1}\right] dy\,dx \\ &= \frac{1}{2\pi\sigma^2}(1-\rho^2)^{-\frac{1}{2}} \int_{-\infty}^{\infty} x \exp - [x^2/2\sigma^2]\ \rho x \sigma \sqrt{1-\rho^2}\sqrt{2\pi}\,dx \\ &= \rho\sigma^2 \end{aligned}$$

ρ is called the correlation of X and Y.

The one-bit techniques are means of estimating this correlation ρ. We define

$$p_{++} = \text{prob}\{X > 0 \text{ and } Y > 0\}$$

and similarly p_{+-}, p_{-+}, and p_{--}.

Let us evaluate p_{++}:

$$p_{++} = \int_0^\infty \int_0^\infty f(x,y)\,dx\,dy$$

$$= \frac{1}{2\pi\sigma^2}(1-\rho^2)^{-\frac{1}{2}} \int_0^\infty \exp - [x^2/2\sigma^2] \int_0^\infty \exp [-(y-\rho x)^2 (2\sigma^2(1-\rho^2))^{-1}\,dy\,dx$$

substituting

$$v = \frac{y-\rho x}{\sqrt{1-\rho^2}}$$

$$p_{++} = \frac{1}{2\pi\sigma^2}(1-\rho^2)^{-\frac{1}{2}} \int_0^\infty \exp - [x^2/2\sigma^2] \sqrt{1-\rho^2} \int_{-\frac{\rho}{\sqrt{1-\rho^2}}x}^\infty \exp - [v^2/2\sigma^2]\,dv\,dx$$

$$= \frac{1}{2\pi\sigma^2} \int_0^\infty \int_{-\frac{\rho}{\sqrt{1-\rho^2}}x}^\infty \exp - [(x^2+v^2)/2\sigma^2]\,dv\,dx$$

Now, changing to the polar coordinates defined by

$$x = r\cos\theta, \quad v = r\sin\theta$$

and changing the element of area from dx dy to r dθ dr

$$p_{++} = (2\pi\sigma^2)^{-1} \int_0^\infty \int_{-\arcsin\rho}^{\frac{\pi}{2}} \exp - [r^2/2\sigma^2]\, r\,d\theta\,dr$$

$$= \frac{1}{2\pi}\left(\frac{\pi}{2} + \arcsin\rho\right) = \frac{1}{4} + \frac{1}{2\pi}\arcsin\rho$$

Since by symmetry,

$$p_{++} + p_{+-} = \frac{1}{2}, \quad p_{+-} = p_{-+}, \text{ and } \quad p_{--} = p_{++},$$

We know p_{+-}, p_{-+}, and p_{--}. We may evaluate

$$p_{++} + p_{--} - p_{+-} - p_{-+} = \frac{2}{\pi}\arcsin\rho$$

Example of application of ρ.

Consider a collection of 2M samples of a Gaussian noise voltage N(t),

$$N_k = N(k\Delta t) \qquad k = -M+1, -M+2, \cdots, M-1, M$$

We then have the spectrum of the noise, F_ℓ, given by the Fourier series relationships

$$F_\ell = \frac{1}{\sqrt{4\pi M}} \sum_{k=-M+1}^{M} N_k \, e^{2\pi i k\ell/2N}$$

$$N_k = \frac{1}{\sqrt{4\pi M}} \sum_{\ell=-M+1}^{M} F_\ell \, e^{-2\pi i k\ell/2N}$$

The radio astronomer is interested in the power spectrum of the noise voltage,

$$P\left(\frac{2\pi\ell}{\Delta t}\right) = |F_\ell|^2,$$

Suppose we have available the autocorrelation function

$$\rho_m = \langle N_k N_{k+m} \rangle$$

$$= \frac{1}{2M} \sum_{k=-M+1}^{M} N_k N_{k+m}$$

Let us Fourier invert ρ_m:

$$\frac{1}{2\pi} \sum_{m=-M+1}^{M} \rho_m \, e^{2\pi i m\ell/2N} = \frac{1}{4\pi M} \sum_{k=-M+1}^{M} N_k \sum_{m=-M+1}^{M} N_{k+m} \, e^{2\pi i m\ell/2N}$$

$$= \frac{1}{4\pi M} \sum_{k=-M+1}^{M} N_k \, e^{-2\pi i k\ell/2N} \sum_{n=-M+1}^{M} N_n \, e^{2\pi i n\ell/2N}$$

$$= |F_\ell|^2$$

[B. Clark - 7/30/69]