Theoretical Analysis of Superconducting Submillimetre-Wave Microstrip Transmission Lines.

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1 Introduction

The behaviour of superconducting microstrip transmission line has been investigated by a number of authors [1, 2]. The techniques used range from conformal transformations through to complete full-wave analyses. Despite this work, it is still difficult to know which approach should be used for designing submillimeter-wave circuits, and indeed the degree to which the various approaches agree. Clearly, every method has its strengths and weaknesses, and to some extent the suitability of a technique depends on the particular application being considered. Clearly, an important consideration is the ease with which a procedure can be used. It might be argued, for example, that full-wave analyses are best, but such calculations require large computers and are slow to use. In many cases simple closed-form expressions may give the required degree of accuracy. Moreover, it is often important to be able to analyse the behaviour of circuits working at frequencies close to the gap and at temperatures approaching the critical temperature; in these cases it is desirable to use a simple perturbation approach which can be integrated with standard methods for characterising the surface behaviour of superconductors.

In this paper, we present a complete design procedure for millimetre wave and submillimetre wave superconducting microstrip transmission lines. We wish to take into account the thickness of the film and the nonuniform current distribution on the strip. In the case where dispersion due to the size of the line is unimportant, a rigorous conformal mapping technique is used. This approach applies to structures much smaller than a wavelength and films that are relatively thick. For large structures where dispersion due to the width of the strip and the inhomogeneous dielectric is important, a spectral-domain approach is effective. This method gives accurate results for the modal dispersion but is normally restricted to thin films. Our treatment, however, allows the thickness of the film to be incorporated. This modification is achieved by defining and then calculating an effective strip width w_l whose value depends on the geometry and in particular on the thickness of the strip conductor. The value of w_l is, in general, different from the effective width w_t used in impedance calculations. The latter is based on fringing considerations and is always greater than the physical width of the strip; w_l, on the other hand, is based on loss calculations and is always less than the physical width of the strip. An important consequence of this modification is that the singularities, which appear in the integrations when calculating the loss, are avoided. A general advantage of our technique is that it can be applied to other planar transmission-line structures such as coplanar waveguides and coupled, overlapping microstrip lines.

2 General TEM analysis

For convenience, we write the series impedance and the shunt admittance of a TEM transmission line in the form [3, 4]

$$Z = j(k_o\eta_o)g_1 + 2g_2Z_s \tag{1}$$

$$Y = j\left(\frac{k_o}{\eta_o}\right)\left(\frac{\epsilon_{fm}}{g_1}\right) \tag{2}$$

where k_o is the free-space wavenumber, $\eta_o = 120\pi$ is the impedance of free space, Z_s is the surface impedance of the conductors, and ϵ_{fm} is the effective dielectric constant—in the modal sense. g_1 and g_2 are geometrical factors, which characterize the particular transmission line being used. Here, we have neglected dielectric loss and assumed that the strip conductor and ground plane are made of the same material; both of these assumptions can be relaxed if required. According to the above formalism we have divided the total field into two parts. The first term in equation (1) characterizes the field external to the conductors and the second term characterizes the field internal to the conductors. Clearly, the challenge is to derive accurate expressions for g_1 and g_2. In the case of a microstrip transmission line, it is convenient to express the geometrical parameters in the following forms:

$$g_1 = \frac{h}{wK_f} \tag{3}$$

$$g_2 = \frac{K_l}{w} = \frac{(F_g + F_s)}{2w} \tag{4}$$

where K_f is the fringing factor, defined as the ratio between the impedance of an ideal parallel-plate waveguide having width w, dielectric thickness h, and no fringing field and that of the actual strip transmission line, both in a homogeneous dielectric. F_s and F_g are the ratios of the power dissipated per unit length in the strip and the ground plane respectively compared to the values obtained for a parallel-plate line having strip width w, zero strip thickness, and a uniform current distribution over the surfaces internal to the waveguide with no current elsewhere. Normally F_s is much greater than F_g due to the current on the strip being confined to a smaller region. Following Pucel et. al. [5] we can write

$$F_s = \frac{\int_c |J_s(\xi)|^2 d\xi}{\left(\int_c |J_s(\xi)| d\xi\right)^2} \tag{5}$$

where $\xi = x/w$. In this expression, $J_s(\xi)$ is the longitudinal current distribution on the surface of the strip, and the integration is carried out over the strip boundaries. An equivalent expression can be written for F_g. We can now see that if we restrict the flow of current to the inner surfaces of the conductors then F_s will be greater than unity because of the additional loss and energy storage which arises from the sharp increase in the current density near the edges. If, however, we allow a significant amount of current to leak to the sides and back of the strip, then F_s may assume values of less than unity. In conclusion, we have reduced the problem of calculating the loss and determining the effect of superconductivity to that of finding K_f, F_g, and F_s.

The complex propagation constant of the transmission line can be determined in the usual way through $\gamma = (ZY)^{1/2} = \alpha + j\beta$. If we apply our expressions for the series impedance and the shunt conductance we find

$$\beta = \beta_m Re \left[1 - \frac{j2\chi Z_s}{k_o \eta_o h}\right]^{1/2} \tag{6}$$

where

$$\chi = K_l K_f \tag{7}$$

and

$$\beta_m = k_o(\epsilon_{fm})^{1/2} \tag{8}$$

is the modal propagation constant. Also, we find for the attenuation constant

$$\alpha = -\beta_m Im \left[1 - \frac{j2\chi Z_s}{k_o \eta_o h}\right]^{1/2} . \tag{9}$$

From the above equations we notice that the complex propagation constant of a microstrip line is increased by the factor in brackets when the line becomes superconducting. This slow-wave effect is due to the kinetic inductance associated with the energy stored in the electron pairs. The dimensions of the microstrip line influence the propagation constant not only through the modal value but also through the effect of the surface impedance; clearly, the geometry effects both the external and internal fields. The modification that results from field penetration is parameterised by the quantity χ, which we shall refer to as the penetration factor. Assuming that the penetration depth is much less than the thickness of the dielectric, χ gives the factor by which the loss of ordinary microstrip is greater than that of an idealised parallel-plate waveguide. χ takes into account the increase in the attenuation constant caused by the current distribution not being uniform and the characteristic impedance being lowered by the fringing field. We will show later that for practical dimensions $0.8 < \chi < 1.5$.

3 The conformal mapping method

It is clear that once K_f and χ are known it is straightforward to calculate the propagation constant. In the case of tiny submillimetre-wave microstrip lines where dispersion due to the dielectric interface can be ignored, but the thickness of the film must be taken into account, it is possible to calculate K_f to high precision by using two conformal transformations. First, a strip of finite thickness in the Z plane is transformed into a slotline with a very-narrow gap and infinitely thin plates of slightly different widths in the W plane. Next, the W plane is transformed into the Z' plane where a parallel-plate waveguide of slightly different plate widths is obtained. This approach was adopted by Chang [2]. Defining

$$p = 2b^2 - 1 + 2b(b^2 - 1)^{1/2} \tag{10}$$

$$b = 1 + t/h \tag{11}$$

where t/h is the thickness of the film divided by the height of the dielectric, we find

$$K_f = \frac{h}{w}\frac{2}{\pi}\ln\left(\frac{2rb}{ra}\right) . \tag{12}$$

ra and rb are respectively the locations of the images, in the W plane, of the centre points on the inner and the outer sides of the strip. Chang provided analytical expressions for ra and rb in terms of the width and thickness of the strip.

Accommodating film thickness when calculating power loss is a notoriously awkward problem. Unfortunately, this calculation is essentially the one we need to do in order to assess the effects of superconductivity. Here we will develop further the conformal mapping technique of Assadourian and Rimai [6]. To calculate the loss we transform the microstrip geometry from the Z plane to the W plane and then perform the appropriate current integration in the W-plane [6]:

$$P = Rs\frac{\epsilon}{\mu}\int |E(W)|^2 \left|\frac{dW}{dZ}\right| |dW| \tag{13}$$

where we require the Schwarz-Christoffel transformation in differential form, which in this case turns out to be

$$\frac{dW}{dZ} = \frac{\pi}{h}p^{1/2}\frac{W}{(W+1)^{1/2}(W+p)^{1/2}} \; . \tag{14}$$

The field in the W plane is simply given by

$$|E(W)| = \left|\frac{E_o h}{\pi W}\right| \tag{15}$$

where E_o is the constant field that would result if the strip where infinitely wide.

These integrals can be evaluated analytically and after some manipulation we find

$$\chi = \frac{Is1 + Is2 + Ig1 + Ig2 + \pi}{2\ln(rb/ra)} \qquad \text{for } w/h < 2 \tag{16}$$

$$\chi = \frac{Is1 + Is2 + Ig1 + Ig2 + \pi}{2\ln(2rb/ra)} \qquad \text{otherwise;} \tag{17}$$

where for the bottom surface of the strip we get

$$\begin{aligned} Is1 &= \ln\left[\frac{2p - (p+1)ra + 2\sqrt{pRa}}{ra(p-1)}\right] \\ Ra &= (1-ra)(p-ra); \end{aligned} \tag{18}$$

for the top surface of the strip we get

$$\begin{aligned} Is2 &= -\ln\left[\frac{(p+1)rb - 2p - 2\sqrt{pRb}}{rb(p-1)}\right] \\ Rb &= (rb-1)(rb-p); \end{aligned} \tag{19}$$

and for the ground plane we have

$$\begin{aligned} Ig1 &= -\ln\left[\frac{(p+1)rb + 2p + 2\sqrt{pRb'}}{rb(p-1)}\right] \\ Rb' &= (rb+1)(rb+p) \; , \end{aligned} \tag{20}$$

and

$$\begin{aligned} Ig2 &= \ln\left[\frac{(p+1)ra + 2p + 2\sqrt{pRa'}}{ra(p-1)}\right] \\ Ra' &= (ra+1)(ra+p)\ . \end{aligned} \tag{21}$$

These expressions are simple to evaluate, and we believe that they are probably the most accurate analytical expressions for calculating the loss of a microstrip line available in the literature.

The above equations are based on the assumption that the field in the W plane is closely related to that of a coaxial line. We can test this assumption by using the second conformal transformation, which maps the slotline in the W plane into a parallel-plate capacitor in the Z' plane. A more accurate expression for the field in the W plane then becomes

$$|E(W)| = \frac{hE_o}{\pi}\left|\frac{2rb}{W(W+2rb)}\right|\ . \tag{22}$$

Using the above expression, we can now calculate the penetration factor:

$$\chi = \frac{1}{2\ln(2rb/ra)}\int_{ra}^{rb} \frac{4rb^2p^{1/2}\,|dW|}{\left|(2rb+W)^2W\,[(W+1)(W+p)]^{1/2}\right|} \tag{23}$$

where the integration is calculated over the ground plane $W = r$ and over the surface of the strip $W = -r$. We have evaluated these integrals, and compared the result, in Fig 1, with the value of χ derived by using the analytical expressions. Clearly, the two calculations agree exceeding well verifying the original assumptions. There is a discrepancy of about 5% at $w/h = 2$, but the two methods become closer both for wider and thinner strips.

In Fig. 1 we also show the value of χ suggested by the recession method of Wheeler [7, 5]. Here the internal inductance is inferred from the external inductance by considering the change in external inductance as the walls of the conductors are moved. In this way, an expression for the attenuation constant of microstrip line was derived by Pucel et.al. [5]. We can use the expressions of Pucel et.al., together with equation (12), to calculate the value of χ indirectly suggested by the recession method. The approach is indirect because the expressions given by Pucel et.al. are presented in terms of the characteristic impedance, and therefore, we must use our value of the fringing factor in order to make the comparison. The conformal mapping, numerical integration, and the recession method agree when the strips are wide. For thin strips the value of χ is greater than unity because of the sharp increase in current at the edges of the strip, whereas for thick strips the value of χ is less than unity because the current flows along the sides of the strip and is more uniformly distributed. There is also good agreement for for thin, narrow strips. In the case of thick, narrow strips, however, there is good agreement for $w/h > 2$ but some discrepancy for $w/h < 2$. This discrepancy can be understood by recalling that in Wheeler's approach the influence of the strip thickness is treated as a perturbation on the strip width.

4 The spectral domain analysis

In the case where dispersion due to the width of the strip and the presence of the dielectric interface cannot be ignored, one of the most accurate techniques for calculating line parameters is the spectral domain method. For example, this situation will occur at millimetre wavelengths when a superconducting line is fabricated on a thick substrate. The spectral domain method is not able, however, in any formal way to calculate the behaviour of strips having finite thickness.

The use of the spectral domain method for calculating power loss in a microstrip line has been described elsewhere [8]. In the context of our work, the calculation of K_f is straightforward, but one has to be more careful when calculating χ. The problem is that the spectral domain method is only formally applicable to cases where the line is infinitely thin, but it is known that the loss in such cases is unbounded because of singularities in the current distribution at the edges of the strip. In real calculations, the number of basis functions used is finite, and therefore a finite value of loss is obtained. Limiting the number of basis functions essentially smooths the behaviour at the edges of the strip, simulating what would be the physical reality [8]. It must be appreciated, however, that this smoothing is a mathematical inadequacy and not something that is central to the model. In this paper we shall employ a method proposed by Lewin [9] which allows the integration over the strip to stop just short of the strip edges. The small distance from the edge is calculated by equating, through a conformal transformation, the loss over a conductor of finite thickness and width w to the loss of an infinitely thin strip having an effective width w_l. Lewin calculated the losses by considering the strips in isolation from the ground planes, that is to say he assumed that the current distributions on the top and bottom surfaces of the strip are equal. This yielded the result $\Delta w = w - w_l = t/(4\pi \exp \pi) \approx t/290$. We, however, employed the equations described in the previous section to obtain an expression for the actual microstrip and obtained a geometry-dependent-correction given by

$$\Delta w = h \frac{\delta^2}{16\pi \exp \pi} \tag{24}$$

where $\delta = p - 1$ and a small t/h ratio was assumed for the sake of simplicity. For $t/h < 0.01$ it is easy to show that $\delta = (8t/h)^{1/2}$ and therefore $\Delta w = \Delta_{Lewin}/2$.

We are now ready to determine χ by using the technique described by Jansen [10] for calculating loss. We can write

$$\chi = 0.5\,[1 + K_f Fs] \tag{25}$$

where F_s is given by

$$F_s = \frac{\int_0^{1-\epsilon_1} |J_s(\xi)|^2 d\xi}{\left(\int_0^{1-\epsilon_1} |J_s(\xi)| d\xi\right)^2} \tag{26}$$

and $\epsilon_1 = 1 - \frac{2\Delta}{w}$.

Spectral domain analysis gives, of course, the current distribution, and after the singularities have been removed we can perform the appropriate integrals to evaluate χ.

In Fig. 2, we show plots of χ against strip width for a microstrip line having a thin metallization ($t/h = 0.0001$) and a substrate dielectric constant of 3.8. We show curves based on the spectral domain method and compare them with our numerical computations. The longitudinal current density over the strip was chosen as

$$J_{zn}(x) = \sum_{i=1}^{n} a_n P_{(i-1)}(\xi) \tag{27}$$

where $P_n(\xi)$ are the Legendre polynomials of order n, and four basis functions were used. With a small number of basis functions precise agreement with the conformal mapping technique is obtained provided that very thin films and low frequencies are considered.

An important advantage of the above choice of basis functions is that the loss calculation yields finite results for any finite value of n. It is only when $n \to \infty$ that the modelled current density becomes singular at the edges. For this reason, the correction given in (24) becomes important only when a large number of basis functions are employed. Another popular choice of basis functions in the spectral domain method is a class of functions that satisfies the edge condition. This choice is made to simulate the behavior of electromagnetic fields near sharp edges in an attempt to speed up the computational efficiency. An example is the distribution

$$J_{zn}(x) = \sum_{i=1}^{n} \frac{\cos[2\pi(i-1)\xi]}{\sqrt{1-\xi^2}} . \tag{28}$$

To carry out loss calculation employing the above current distribution, the correction given by (24) becomes necessary.

The ability of the spectral domain method to handle dispersion is illustrated in Fig. 2 where we show the behaviour when $\lambda_o/h = 12$. Although the corresponding frequency is not particularly high, dispersion in χ can readily be seen. For narrow strips, dispersion in χ is caused primarily by dispersion in K_f whereas for wide strips it is caused by dispersion in both K_f and F_s.

The ability of the spectral domain method to predict loss has been investigated experimentally [8]. Generally, it seems that it should be possible to use the spectral domain method to calculate the behaviour of highly-dispersive superconducting circuits having multiple coupled strips and multiple dielectrics.

5 Superconducting microstrip

Having established the integrity of the equations based on conformal transformations, we can now apply the technique to superconducting lines. For a strip that has a thickness that is several times the penetration depth λ and that is at a temperature well below the critical temperature and that is operating at a frequency well below the gap, the surface impedance is given by

$$Z_s = j\omega\mu_o\lambda = jk_o\eta_o\lambda . \tag{29}$$

We can substitute this expression into the equations for the propagation constant to find

$$\beta = \beta_m[1 + 2\chi\frac{\lambda}{h}]^{1/2} . \tag{30}$$

In Fig. 3, we show the superconducting effective dielectric constant of a typical superconducting Nb line as a function of line width. More specifically, we show the effect of choosing $\chi = 1$, and $\chi \neq 1$, and compare the results with the behaviour of a non-superconducting line. In the case of the superconducting line, it can be seen how the nonuniform the current distribution $\chi \neq 1$ modifies the slow-wave effect.

To calculate the loss in a line operating at temperatures close to the critical temperature or frequencies close to the gap, it is necessary to use BCS [11] theory when calculating the surface impedance. The surface impedance is complex and is given by [1]

$$Z_s = \left(\frac{j\omega\mu_o}{\sigma}\right)^{1/2} \coth\left[(j\omega\mu_o\sigma)^{1/2} t\right] . \tag{31}$$

Finally, in Figs. 4 we show the loss of a typical Nb line operating at frequencies in the submillimetre-wave range with $\chi = 0.88$. In Fig. 4, we compare the loss of the line at 5K and the loss of the line at 2K. It is clear that at 5K and for frequencies above 500GHz, the loss in the line is not negligible. For example, the loss in the line at 500GHz is 0.5db/mm and approaches 1dB/mm at 700GHz. If, however, we cool to the line to 2K, the loss is reduced by a factor of 250.

References

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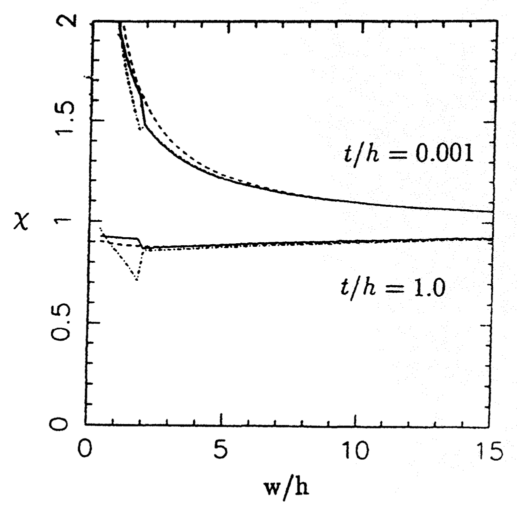

Figure 1: The penetration factor is shown as a function of the normalized strip width for two normalized strip thicknesses: $t/h = 1.0$ and 0.001. The full lines were calculated using the analytical expressions of the conformal mapping technique, the broken lines were calculated using numerical integration and the chain lines were calculated using the recession method.

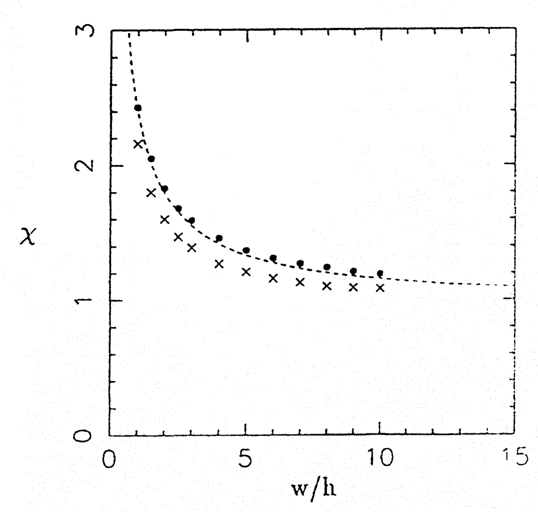

Figure 2: The penetration factor is shown as a function of the normalized strip width for $t/h = 0.0001$. The broken line was calculated using numerical integration of the expression derived through the conformal mapping technique. The dots show points based on the spectral domain method when the wavelength of operation λ_o is large. The crosses show the dispersive case when $\lambda_o/h = 12$.

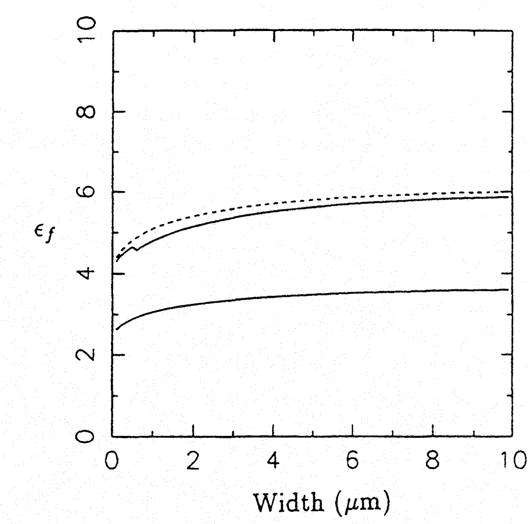

Figure 3: The overall effective dielectric constant, ϵ_f, of a typical Nb sub–millimeter–wave superconducting microstrip transmission line. We have taken $h = 300$ nm, $t = 300 nm \epsilon = 3.8$ and $\lambda = 0.1$ nm. The lower full line shows the case when the line is not superconducting. The upper full line shows the case when the line is superconducting and the fringing and penetration factors are taken into account. The broken line shows the case when the penetration factor is set equal to unity, $\chi = 1$. These curves were based on the analytical expressions of the conformal mapping technique.

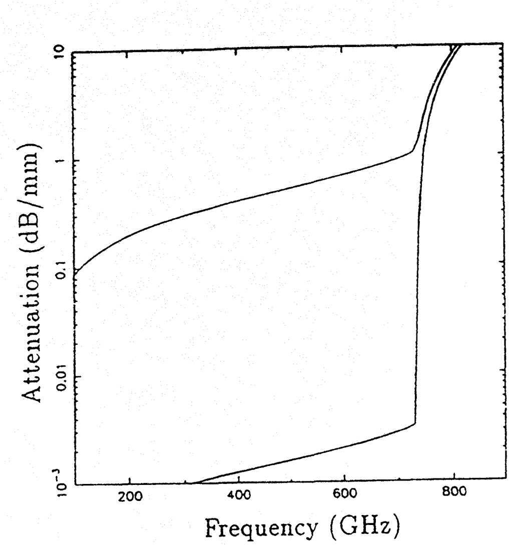

Figure 4: The loss in a typical Nb superconducting microstrip line as a function of frequency . We have taken $w = 750$nm, $t = 300$nm, $\lambda = 0.1$ nm and $\epsilon = 3.8$. The modal effective dielectric constant becomes $\epsilon_{fm} = 2.6$. The full upper line is for a temperature of 5 K and the lower line is for a temperature of 2 K. The plots were calculated using the analytical expressions of the conformal mapping technique.