Transmission and Reflection Characteristics of Slightly Irregular Wire-Grids for Arbitrary Angles of Incidence and Grid Rotation

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1 Introduction

Although there has been many different theoretical studies developed for calculating the transmission and reflection characteristics of freestanding wire-grids, most of them, to the authors' knowledge, were approximate ones only applicable to the cases where the diameter of the wire is much smaller than the wavelength or the spacing between wires [1][2]. It was only rather recently that more exact treatment of wire-grid without such approximations became possible by the Green-function method [3] or the latticesum method [4]. In the first half of this paper, a method for calculating transmission and reflection characteristics of wire-grids consisting of a periodic array of cylinders for arbitrary directions of incidence and grid rotation is presented by extending a very accurate and efficient calculation method for wire-grids based on the latticesum method proposed by Yasumoto *et al.*[5].

In most of the theories which include the one presented in the first half of this paper, the wire-grid has been assumed to be a periodic array of parallel cylindrical wires with a constant spacing between them. In actual wire-grids used in millimeter- and submillimeter-wave regions, the grids are never free from irregularity in wire spacing due to the difficulty in producing evenly spaced grids as the spacing decreases [6][7]. There have been almost no detailed theoretical studies made about the effects of grid imperfection on its performance except the one by Houde *et al.* [8].

In the latter half of this paper, a perturbation theory for calculating the characteristics of wire-grid with slight random irregularity in its spacing is proposed. In this theory, the irregularity in wire-grid is modeled as random errors in wire positions from regular positions, and is treated as a perturbation to the exact theory of periodic wire-grid by assuming that the positional errors of wires are zero-mean statistically uncorrelated independent variables from wire to wire. In order to validate the applicability of the proposed theory for actual irregular wire-grid, results of measurements made for actual wire-grids with different degrees of irregularity at millimeter and submillimeter wavelengths are compared with those of the theoretical calculations.

2 Definition of the Coordinate System

In this paper, we define the coordinate systems as shown in Fig. 1 relating the geometry of the wire-grid and the incident wave. The wire-grid is a periodic array of parallel wires of radius a with a period h. In the x-y-z coordinate system, the plane of wire-grid lies in the x-z plane. Each wire supposed to be an infinitely long cylinder is directed along the z-axis. The plane of incidence in which the incident wavenumber vector \mathbf{k}_0

lies is defined as the X-Y plane $(Y \equiv y)$ which is made by rotating the x-y plane by ϕ_q around the y-axis. We refer to the rotation angle ϕ_q as the grid rotation angle.

In reference to the X-Y-Z coordinate system, the incident wavenumber vector \mathbf{k}_0 lying in the X-Y plane is given by $\mathbf{k}_0 = -\mathbf{i}_X k_0 \sin \chi - \mathbf{i}_Y k_0 \cos \chi$ where χ is the angle of incidence. If we define the angles θ_{in} and ϕ_{in} as in Fig. 1 such that the incident wavenumber vector \mathbf{k}_0 is expressed in the (x, y, z) coordinate system as

$$\boldsymbol{k}_0 = -k_0 \sin \theta_{in} (\cos \phi_{in} \boldsymbol{i}_x + \sin \phi_{in} \boldsymbol{i}_y) + k_0 \cos \theta_{in} \boldsymbol{i}_z.$$

the relationship between (χ, ϕ_g) and (θ_{in}, ϕ_{in}) can be written as

$$\theta_{in} = \cos^{-1}(\sin\chi\sin\phi_g).$$

$$\phi_{in} = \tan^{-1}(\cot\chi\sec\phi_g).$$





3 T Matrix of Isolated Cylindrical Wire for Arbitrary Angle of Incidence

Before considering an array of cylinders. let us consider the scattering of a plane wave by a single isolated cylinder whose axis coincides with the z-axis in Fig 1.

In order for the theory to be applicable not only to metallic wires but also to dielectric cylinders, we assume that the material of wire is a lossy dielectrics characterized by a relative complex permittivity ϵ_r . When the material of the wire is a metallic conductor with a finite conductivity σ , the relative complex permittivity given by $\epsilon_r = 1 + j\sigma/(\omega\epsilon_0)$ should be substituted¹.

3.1 Incident Wave

For later discussions, let us decompose the field of the incident wave into the TM (transverse magnetic) and TE wave components whose electric field are polarized in a plane parallel and perpendicular to the direction of wire. respectively.

For the case of TM incident wave, the incident field at (ρ_0, ϕ_0, z) in the cylindrical coordinate system can be generated by the electric Hertz potential [9] given by

$$\Pi_{zi} = \sum_{n=-\infty}^{\infty} A_i e^{jk_0 z \cos\theta_{in}} J_n(k_0 \rho_0 \sin\theta_{in}) e^{jn(\phi_0 - \phi_{in} - \pi + \frac{\pi}{2})} = e^{jk_0 z \cos\theta_{in}} \boldsymbol{P}_0^T \cdot \boldsymbol{a}^{in} \quad (4)$$

where $\mathbf{P}_0 = [J_m(k_0\rho_0\sin\theta_{in})e^{jm\phi_0}]$ and \mathbf{a}^{in} is the *TM*-incident-wave amplitude vector defined as $\mathbf{a}^{in} = A_i[(-j)^m e^{-jm\phi_{in}}]$, and A_i is the incident field amplitude related with the incident electric field E_0 by $A_i = E_0/(k_0^2\sin\theta_{in})$.

On the other hand, for the case of the TE incident wave, the incident field can be generated by the electric Hertz potential given by

$$\Pi_{mzi} = e^{jk_0 z \cos \theta_{in}} \boldsymbol{P}_0^T \cdot \boldsymbol{b}^{in}, \qquad (5)$$

where \boldsymbol{b}^{in} is the *TE*-incident-wave amplitude vector defined as $\boldsymbol{b}^{in} = B_i[(-j)^m e^{-jm\phi_{in}}]$, and B_i is the incident field amplitude related with the incident electric field E_0 by $B_i = E_0/(\eta k_0^2 \sin \theta_{in})$ where $\eta(=\sqrt{\mu_0/\epsilon_0})$ is the impedance of free space.

¹The time dependence of field is assumed to be $\exp(-j\omega t)$ and is omitted throughout this paper.

3.2 Scattered Wave

We express the TM and TE components of the scattered wave by the electric Hertz potential Π_{zs} and the magnetic Hertz potential Π_{mzs} , respectively, by using unknown coefficient A_{sn} and B_{sn} as

$$\Pi_{zs} = e^{jk_0 z \cos \theta_{in}} \boldsymbol{Q}_0^T \cdot \boldsymbol{a}_0^{sc}, \quad \text{and} \quad \Pi_{mzs} = e^{jk_0 z \cos \theta_{in}} \boldsymbol{Q}_0^T \cdot \boldsymbol{b}_0^{sc}, \tag{6}$$

where $\boldsymbol{a}_{0}^{sc} = [(-j)^{m} A_{sm} e^{-jm\phi_{in}}], \boldsymbol{b}_{0}^{sc} = [(-j)^{m} B_{sm} e^{-jm\phi_{in}}] \text{ and } \boldsymbol{Q}_{0} = [H_{m}^{(1)}(k_{0}\rho_{0}\sin\theta_{in})e^{jm\phi_{0}}].$

3.3 Field Inside the Wire

Inside the wire, we express the electric Hertz potential Π_{ze} and the magnetic Hertz potential Π_{mze} using the unknown coefficients A_{en} and B_{en} , respectively, as

$$\Pi_{ze} = \sum_{n=-\infty}^{\infty} A_{en} e^{jk_0 z \cos \theta_{in}} J_n(k_1 \rho_0 \sin \theta_1) e^{jn(\phi_0 - \phi_{in} - \pi + \frac{\pi}{2})},$$
(7)

$$\Pi_{mze} = \sum_{n=-\infty}^{\infty} B_{en} e^{jk_0 z \cos\theta_{in}} J_n(k_1 \rho_0 \sin\theta_1) e^{jn(\phi_0 - \phi_{in} - \pi + \frac{\pi}{2})},$$
(8)

where $k_1 = \sqrt{\epsilon_r} k_0$ and $k_1 \sin \theta_1 = \sqrt{k_1^2 - k_0^2 \cos^2 \theta_{in}}$.

3.4 T-Matrix for Isolated Cylinder

If we define the T-matrix [9] relating the m-th mode of the scattered amplitude with the incident amplitude as

$$\begin{pmatrix} A_{sm} \\ B_{sm} \end{pmatrix} = \begin{pmatrix} T_{AA}^{(m)} & T_{AB}^{(m)} \\ T_{BA}^{(m)} & T_{BB}^{(m)} \end{pmatrix} \begin{pmatrix} A_i \\ B_i \end{pmatrix},$$
(9)

the the elements of the T-matrix are derived from the boundary conditions on the surface of cylinder for each modes as [9]:

$$\begin{pmatrix} T_{AA}^{(m)} & T_{AB}^{(m)} \\ T_{BA}^{(m)} & T_{BB}^{(m)} \end{pmatrix} = \boldsymbol{U}\boldsymbol{M}^{(m)-1}\boldsymbol{N}^{(m)},$$
(10)

where the elements of the 2×4 matrix \boldsymbol{U} , the 4×4 matrix $\boldsymbol{M}^{(m)}$, and the 4×2 matrix $\boldsymbol{N}^{(m)}$ are given as:

$$U_{11} = U_{23} = 1$$
, $U_{12} = U_{13} = U_{14} = U_{21} = U_{22} = U_{24} = 0$, and (11)

$$\begin{split} &M_{11}^{(m)} &= M_{23}^{(m)} = k_0^2 \sin^2 \theta_{in} H_m^{(1)}(k_0 a \sin \theta_{in}), & M_{41}^{(m)} &= j\omega\epsilon_0 k_0 \sin \theta_{in} H_m^{(1)'}(k_0 a \sin \theta_{in}), \\ &M_{12}^{(m)} &= M_{24}^{(m)} = -k_1^2 \sin^2 \theta_1 J_m(k_1 a \sin \theta_1), & M_{42}^{(m)} &= -j\omega\epsilon_r\epsilon_0 k_1 \sin \theta_1 J_m'(k_1 a \sin \theta_1), \\ &M_{13}^{(m)} &= M_{14}^{(m)} = M_{21}^{(m)} = M_{22}^{(m)} = 0, & N_{11}^{(m)} &= N_{22}^{(m)} = -k_0^2 \sin^2 \theta_{in} J_m(k_0 a \sin \theta_{in}), \\ &M_{31}^{(m)} &= M_{43}^{(m)} = -\frac{k_0 m \cos \theta_{in}}{a} H_m^{(1)}(k_0 a \sin \theta_{in}), & N_{12}^{(m)} &= N_{21}^{(m)} = 0, \\ &M_{32}^{(m)} &= M_{44}^{(m)} = \frac{k_0 m \cos \theta_{in}}{a} J_m(k_1 a \sin \theta_1), & N_{31}^{(m)} &= N_{42}^{(m)} = \frac{k_0 m \cos \theta_{in}}{a} J_m(k_0 a \sin \theta_{in}), \\ &M_{33}^{(m)} &= -j\omega\mu_0 k_0 \sin \theta_{in} H_m^{(1)'}(k_0 a \sin \theta_{in}), & N_{41}^{(m)} &= -j\omega\epsilon_0 k_0 \sin \theta_0 J_m'(k_0 a \sin \theta_{in}), \\ &M_{34}^{(m)} &= j\omega\mu_0 k_1 \sin \theta_1 J_m'(k_1 a \sin \theta_1). & N_{41}^{(m)} &= -j\omega\epsilon_0 k_0 \sin \theta_0 J_m'(k_0 a \sin \theta_{in}). \end{split}$$

For the discussion that follows, if we define the incident amplitude vector $\boldsymbol{\alpha}^m$ and the scattered amplitude vector $\boldsymbol{\alpha}_0^{sc}$ as

$$\boldsymbol{\alpha}^{in} = \begin{pmatrix} \boldsymbol{a}^{in} \\ \boldsymbol{b}^{in} \end{pmatrix}, \quad \boldsymbol{\alpha}_0^{sc} = \begin{pmatrix} \boldsymbol{a}_0^{sc} \\ \boldsymbol{b}_0^{sc} \end{pmatrix}.$$
(12)

and the *T*-matrix $\boldsymbol{\Upsilon}$ relating $\boldsymbol{\alpha}_{0}^{sc}$ with $\boldsymbol{\alpha}^{in}$ as

$$\boldsymbol{\alpha}_{0}^{sc} = \boldsymbol{\Upsilon} \ \boldsymbol{\alpha}^{in}, \quad \text{where} \quad \boldsymbol{\Upsilon} = \begin{pmatrix} \boldsymbol{T}^{(AA)} & \boldsymbol{T}^{(AB)} \\ \boldsymbol{T}^{(BA)} & \boldsymbol{T}^{(BB)} \end{pmatrix}.$$
 (13)

the elements of Υ are given by using the elements of the *T*-matrix for the *m*-th mode given in (10) as

$$T_{mn}^{(AA)} = \delta_{mn} T_{AA}^{(m)}, \quad T_{mn}^{(AB)} = \delta_{mn} T_{AB}^{(m)}, \quad T_{mn}^{(BA)} = \delta_{mn} T_{BA}^{(m)}, \quad T_{mn}^{(BB)} = \delta_{mn} T_{BB}^{(m)}.$$
(14)

As is found from (10), for cylinders with finite conductivity, there exists coupling between the TM and TE components when $\theta_{in} \neq \pi/2$.

4 Scattering of Plane Wave by a Regular Wire-Grid for Arbitrary Angle of Incidence and Grid Rotation

Before considering irregular wire-grids, let us consider a regular wire-grid of a periodic array of equally spaced cylinders with a spacing of h as shown in Fig. 1. In this case, the x-y dependence $\Psi^{sc}(x,y)$ of the scattered field

$$\Psi^{sc}(x,y,z) = e^{jk_0 z \cos\theta_{in}} \Psi^{sc}(x,y) \tag{15}$$

at a point (x, y, z) outside the cylinders is given by using Floquet's theorem as

$$\Psi^{sc}(x,y) = \sum_{l=-\infty}^{\infty} \boldsymbol{Q}_l^T \cdot \boldsymbol{a}_0^{sc} e^{-jlk_0 h \sin\theta_{in} \cos\phi_{in}}, \qquad (16)$$

where $Q_l = [H_n^{(1)}(k_0\rho_l\sin\theta_{in})e^{jn\phi_l}]$ with $\rho_l = \sqrt{(x-lh)^2 + y^2}$ and $\phi_l = \cos^{-1}[(x-lh)/\rho_l]$. By using Gegenbauer's addition theorem of Hankel functions given as

$$H_n^{(1)}(k_0\rho_l\sin\theta_{in})e^{jn\phi_l} = \sum_{m=-\infty}^{\infty} J_m(k_0\rho_0\sin\theta_{in})e^{jm\phi_0}H_{n-m}^{(1)}(k_0hl\sin\theta_{in}).$$
 (17)

 \boldsymbol{Q}_l is expressed as

$$\boldsymbol{Q}_{l}^{T} = \boldsymbol{P}_{0}^{T} \cdot \boldsymbol{\xi}_{l} \qquad (l \neq 0)$$
(18)

where $\xi_{l,mn} = H_{n-m}^{(1)}(k_0 lh \sin \theta_{in}).$

The x-y dependence $\Psi(x,y)$ of the total field $\Psi(x,y,z) = \Psi(x,y)e^{jk_0z\cos\theta_{in}}$ is given by

$$\Psi(x,y) = \boldsymbol{P}_0^T \cdot (\boldsymbol{a}^{in} + \boldsymbol{L} \cdot \boldsymbol{a}_0^{sc}) + \boldsymbol{Q}_0^T \cdot \boldsymbol{a}_0^{sc}.$$
(19)

The first and the second terms of the right-hand side of (19) are regarded as the field incident on the 0-th wire and the field scattered by the 0-th wire, respectively.

By considering the coupling between the TM and TE components given in (9), the scattered field amplitude α_0^{sc} can be expressed

$$\boldsymbol{\alpha}_{0}^{sc} = \boldsymbol{\Upsilon} \cdot (\boldsymbol{\alpha}^{in} + \boldsymbol{\Lambda} \cdot \boldsymbol{\alpha}_{0}^{sc}) \tag{20}$$

by using the *T*-matrix Υ for isolated single wire. In (20), Λ is the lattice-sum matrix given as follows:

$$\mathbf{\Lambda} = \begin{pmatrix} \mathbf{L} & 0\\ 0 & \mathbf{L} \end{pmatrix},\tag{21}$$

where the elements of L are given by the lattice sum as

$$L_{m,n} = \sum_{l=-\infty, l\neq 0}^{\infty} H_{m-n}^{(1)} (lk_0 h \sin \theta_{in}) e^{-jlk_0 h \sin \theta_{in} \cos \phi_{in}}.$$
 (22)

By solving (20) in terms of α_0^{sc} , we obtain the scattered amplitude as

$$\boldsymbol{\alpha}_0^{sc} = (\boldsymbol{I} - \boldsymbol{\Upsilon} \cdot \boldsymbol{\Lambda})^{-1} \cdot \boldsymbol{\Upsilon} \cdot \boldsymbol{\alpha}^{in}.$$
(23)

The sum of the infinite series in the right-hand side of (22), which is usually referred to as the lattice sum and is notorious for its desperately slow convergence in problems of periodic Green's functions, can be numerically calculated by an accurate and efficient method based on the Fourier integral representation of Hankel functions proposed by Yasumoto *et al.*[5] and [4].

4.1 Reflected and Transmitted Waves

From the scattered amplitude α_0^{sc} obtained above, the TM components of the reflected wave Ψ^r and the transmitted waves Ψ^t can be obtained as follows:

$$\Psi_{TM}^r(x,y,z) = \sum_{\nu=-\infty}^{\infty} \boldsymbol{p}_{\nu}^T \cdot \boldsymbol{a}_0^{sc} e^{j(k_{x\nu}x + k_{y\nu}y + k_0z\cos\theta_{in})}$$
(24)

$$\Psi_{TM}^{t}(x, y, z) = \sum_{\nu = -\infty}^{\infty} (\delta_{\nu 0} + \boldsymbol{q}_{\nu}^{T} \cdot \boldsymbol{a}_{0}^{sc}) e^{j(k_{x\nu}x - k_{y\nu}y + k_{0}z\cos\theta_{in})}$$
(25)

where

$$\boldsymbol{p}_{\nu} = \begin{bmatrix} \frac{2(-j)^{m}(k_{x\nu}+jk_{y\nu})^{m}}{hk_{y\nu}k_{0}^{m}\sin^{m}\theta_{in}} & (m \ge 0) \\ \frac{2j^{|m|}(k_{x\nu}-ik_{y\nu})^{|m|}}{hk_{y\nu}k_{0}^{|m|}\sin^{|m|}\theta_{in}} & (m < 0) \end{bmatrix} \quad \text{and} \quad \boldsymbol{q}_{\nu} = \begin{bmatrix} \frac{2(-j)^{m}(k_{x\nu}-ik_{y\nu})^{m}}{hk_{y\nu}k_{0}^{m}\sin^{m}\theta_{in}} & (m \ge 0) \\ \frac{2j^{|m|}(k_{x\nu}+ik_{y\nu})^{|m|}}{hk_{y\nu}k_{0}^{|m|}\sin^{|m|}\theta_{in}} & (m < 0) \end{bmatrix}$$

with $k_{x\nu} = \frac{2\nu\pi}{h} - k_0 \sin\theta_{in} \cos\phi_{in}$ and $k_{y\nu} = \sqrt{k_0^2 \sin^2\theta_{in} - k_{x\nu}^2}$. The *TE* components of the reflected wave Ψ^r and the transmitted waves Ψ^t can be obtained similarly just by replacing \boldsymbol{a}_0^{sc} with \boldsymbol{b}_0^{sc} in (24) and (25).

It should be noted that, if $h|\sin\theta_{in}|(1+\cos\phi_{in}) < \lambda$, only the modes with $\nu = 0$ of the transmitted and reflected waves are propagated, and other modes with $\nu \neq 0$ are

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evanescent. In this particular case, the power reflection coefficient R and the power transmission coefficient T of the grid are given as

$$R_{TM} = |\boldsymbol{p}_0^T \cdot \boldsymbol{a}_0^{sc}|^2, \quad \text{and} \quad T_{TM} = |1 - \boldsymbol{q}_0^T \cdot \boldsymbol{a}_0^{sc}|^2.$$
(26)

for the TM components of reflected and transmitted waves, respectively. The power reflection coefficient R_{TE} and the power transmission coefficient T_{TE} for the TE component can be obtained similarly by replacing a_0^{sc} with b_0^{sc} in (26).

You can try calculations of regular grids on our web site [10].

5 Perturbation Theory for Slightly Irregular Wire-Grids

So far, we have assumed the wire-grid as a perfectly periodic array of equally spaced parallel wires. In this section, we consider an irregular wire-grid whose wire spacing is not perfectly uniform with slight random displacement of each wire position from its nominal position keeping the parallelism with each other. Although there might be other types of irregularity in wire-grids such as the nonuniformity in wire diameter or the nonflatness of the grid plane. experience shows that they are less significant as compared with the nonuniformity in wire spacing considering the usual fabrication method in which wire is wound on a precisely fabricated metallic frame. We, therefore, focus on the effects of irregularity in wire spacing ignoring other irregularities.

Let the random irregularity in wire spacing of a wire-grid be given by a column vector $\boldsymbol{\eta} = (\cdots, \eta_{-1}, \eta_0, \eta_1, \cdots, \eta_l, \cdots)^T$ where η_l is a random error in the *x* position of the center of the *l*-th wire from the right position *lh* as shown in Fig. 2. The scattering amplitude of the 0th wire given by (23) is modified for this irregular grid can be expressed as

$$\boldsymbol{\alpha}_{0}^{SC}(\boldsymbol{\eta}) = (\boldsymbol{I} - \boldsymbol{\Upsilon} \cdot \boldsymbol{\Lambda}(\boldsymbol{\eta}))^{-1} \cdot \boldsymbol{\Upsilon} \cdot \boldsymbol{\alpha}^{in}, \qquad (27)$$

where $\Lambda(\boldsymbol{\eta})$ is the lattice-sum matrix for irregular grid given by replacing lh in the lattice sum matrix given by (21) through (22) with $lh + \eta_l$. If we assume the positional errors are small variables such that $\eta_l \ll h$ $(l = \cdots - 1, 0, 1, \cdots)$, $\boldsymbol{\alpha}_0^{SC}(\boldsymbol{\eta})$ can be approximated to the second order of $\boldsymbol{\eta}$ as

$$\boldsymbol{\alpha}_{0}^{SC}(\boldsymbol{\eta}) \simeq \boldsymbol{\alpha}_{0}^{SC}(\boldsymbol{\eta}=0) + \sum_{l=-\infty}^{\infty} \frac{\partial \boldsymbol{\alpha}_{0}^{SC}(\boldsymbol{\eta})}{\partial \eta_{l}} \Big|_{\boldsymbol{\eta}=0} \eta_{l} + \frac{1}{2} \sum_{l=-\infty}^{\infty} \frac{\partial^{2} \boldsymbol{\alpha}_{0}^{SC}(\boldsymbol{\eta})}{\partial \eta_{l}^{2}} \Big|_{\boldsymbol{\eta}=0} \eta_{l}^{2}.$$
(28)

where the partial derivatives in the right-hand side of (28) can be obtained as follows after some algebra:

$$\frac{\partial \boldsymbol{\alpha}_{0}^{SC}(\boldsymbol{\eta})}{\partial \eta_{l}} = (\boldsymbol{I} - \boldsymbol{\Upsilon} \boldsymbol{\Lambda}(\boldsymbol{\eta}))^{-1} \boldsymbol{\Upsilon} \frac{\partial \boldsymbol{\Lambda}}{\partial \eta_{l}} \boldsymbol{\alpha}_{0}^{SC}(\boldsymbol{\eta}), \qquad (29)$$

Figure 2: Irregular wire-grid with random displacements of grid positions

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$$\frac{\partial^2 \boldsymbol{\alpha}_0^{SC}(\boldsymbol{\eta})}{\partial \eta_l^2} = \left[(\boldsymbol{I} - \boldsymbol{\Upsilon} \boldsymbol{\Lambda}(\boldsymbol{\eta}))^{-1} \boldsymbol{\Upsilon} \frac{\partial^2 \boldsymbol{\Lambda}}{\partial \eta_l^2} + 2 \left\{ (\boldsymbol{I} - \boldsymbol{\Upsilon} \boldsymbol{\Lambda}(\boldsymbol{\eta}))^{-1} \boldsymbol{\Upsilon} \frac{\partial \boldsymbol{\Lambda}}{\partial \eta_l} \right\}^2 \right] \cdot \boldsymbol{\alpha}_0^{SC}(\boldsymbol{\eta}). \quad (30)$$

If we assume that the positional errors of wires $(\dots, \eta_{-1}, \eta_0, \eta_1, \dots, \eta_l, \dots)$ are zeromean statistically independent variables from wire to wire, we obtain the ensemble average of the scattering amplitude $\alpha_0^{SC}(\eta)$ as

$$\overline{\boldsymbol{\alpha}_{0}^{SC}(\boldsymbol{\eta})} \simeq \boldsymbol{\alpha}_{0}^{SC}(0) + \frac{1}{2} \sum_{l=-\infty}^{\infty} \frac{\partial^{2} \boldsymbol{\alpha}_{0}^{SC}(\boldsymbol{\eta})}{\partial \eta_{l}^{2}} \Big|_{\boldsymbol{\eta}=0} \overline{\boldsymbol{\eta}^{2}},$$
(31)

where $\alpha_0^{SC}(0) \equiv \alpha_0^{SC}(\eta = 0)$, and $\overline{\eta^2}$ is the variance of the random positional errors of wires from their nominal positions.

By substituting (30) into (31), we obtain

$$\overline{\boldsymbol{\alpha}_{0}^{SC}(\boldsymbol{\eta})} \simeq \left[1 + \frac{\overline{\eta^{2}}}{2} \left[(\boldsymbol{I} - \boldsymbol{\Upsilon}\boldsymbol{\Lambda}(0))^{-1} \boldsymbol{\Upsilon} \sum_{l=-\infty}^{\infty} \frac{\partial^{2}\boldsymbol{\Lambda}(\boldsymbol{\eta})}{\partial \eta_{l}^{2}} \Big|_{\boldsymbol{\eta}=0} \right] + 2 \sum_{l=-\infty}^{\infty} \left\{ (\boldsymbol{I} - \boldsymbol{\Upsilon}\boldsymbol{\Lambda}(0))^{-1} \boldsymbol{\Upsilon} \frac{\partial \boldsymbol{\Lambda}(\boldsymbol{\eta})}{\partial \eta_{l}} \Big|_{\boldsymbol{\eta}=0} \right\}^{2} \right] \boldsymbol{\alpha}_{0}^{SC}(0).$$
(32)

In (32), the partial derivatives are taken at $\eta = 0$, and $\Lambda(\eta)$ is given by

$$\mathbf{\Lambda}(\boldsymbol{\eta}) = \begin{pmatrix} \mathbf{L}(\boldsymbol{\eta}) & 0\\ 0 & \mathbf{L}(\boldsymbol{\eta}) \end{pmatrix}, \tag{33}$$

where $\boldsymbol{L}(\boldsymbol{\eta})$ is given by

$$L_{m,n}(\boldsymbol{\eta}) = S_{m-n}(\boldsymbol{\eta}) = \sum_{l=-\infty, l\neq 0}^{\infty} H_{m-n}^{(1)}(k_0(lh+\eta_l)\sin\theta_{in})e^{-jk_0(lh+\eta_l)\sin\theta_{in}\cos\phi_{in}}.$$
 (34)

With the aid of the recurrence formula and the Fourier integral representation of the Hankel functions as in [5], we can calculate the partial derivatives of the components of the lattice-sum matrix by using

$$\frac{\partial S_n}{\partial \eta_l}\Big|_{\boldsymbol{\eta}=0} = (-1)^n \frac{j e^{jk_x lh}}{k_0^n \sin^n \theta_{in}} \times \int_{-\infty}^{\infty} \frac{[\xi + j\kappa(\xi)]^m [k_x + \kappa(\xi)]}{\kappa(\xi)} e^{j\kappa(\xi) lh} d\xi, \quad (35)$$

and

$$\sum_{l=-\infty}^{\infty} \frac{\partial^2 S_n}{\partial \eta_l^2} \Big|_{\boldsymbol{\eta}=0} = (-1)^{n+1} \frac{e^{jk_x h}}{\pi k_0^n \sin^n \theta_{in}} \times \int_{-\infty}^{\infty} \frac{[\xi + j\kappa(\xi)]^m [k_x + \kappa(\xi)]^2}{\kappa(\xi)} \frac{e^{j\kappa(\xi)h}}{1 - e^{j[k_x + \kappa(\xi)]h}} d\xi,$$
(36)

where $k_x = -k_0 \sin \theta_{in} \cos \phi_{in}$ and

$$\kappa(\xi) = \begin{cases} \sqrt{k_0^2 \sin^2 \theta_{in} - \xi^2} & \text{for } k_0 \sin \theta_{in} \ge |\xi|, \\ j\sqrt{\xi^2 - k_0^2 \sin^2 \theta_{in}} & \text{for } k_0 \sin \theta_{in} < |\xi|. \end{cases}$$
(37)

Once the scattered amplitude vector $\overline{\alpha_0^{sc}}$ is obtained from (32), the reflection and transmission coefficient of the irregular wire-grid can be obtained from (26), and corresponding equations for the *TE* components with (32) and (12).

6 Comparison of Numerical Calculations with Measurements

To validate the effectiveness of the proposed theory in estimating the characteristics of practical freestanding wire-grids. results of numerical calculations are compared with results of measurements made for actual wire-grids with different irregularities.

The conditions for the numerical calculations and the measurements were as follows:

- the material of wires: tungsten ($\sigma^{-1} = 5.5 \times 10^{-8} \Omega m \ [6]$).
- the wire diameter: $2a = 10 \ \mu m$, the nominal spacing between wires: $h = 25 \ \mu m$.
- the angle of incidence: $\chi = 45^{\circ}$.

Measurements were made for two wire-grids with different degree of irregularity. We, hereafter, refer to the one with less irregularity as Grid #1 and to the other as Grid #2. Figures 3 show the microscopic images of the grid planes of Grid #1 and #2 along with their probabity distributions of the displacements of wires from nominal positions. From Figs. 3(b1) and (b2), the standard deviations $\sigma_{\eta}(=\sqrt{\overline{\eta^2}})$ of the displacements of wires from their nominal positions are estimated to be 22.7 % and 52 %, respectively, of the nominal wire spacing($h = 25\mu$ m).

Figs. 4(a) and (b) show the transmission and reflection coefficients. respectively. for the case of *TE*-wave incidence where the incident electric field is orthogonal to the direction of wires. The solid curves show the characteristics calculated for ideal grid without irregularity, and the dashed, dash-dot. and dotted curves show the characteristics calculated for irregular grids whose irregularities σ_{η} are assumed to be 20% ($\sigma_{\eta} = 0.20 h$), 40% ($\sigma_{\eta} = 0.40 h$), and 50 % ($\sigma_{\eta} = 0.50 h$) of the nominal wire spacing. Calculated results are shown for two different grid rotation angles. $\phi_g = 0^{\circ}$ (denoted as G_{perp}) and 90° (denoted as G_{para}), by thin and thick curves, respectively. In these Figures, E_{perp} and E_{para} denote the characteristics for the transmitted or reflected electric field component perpendicular and parallel to the plane of incidence.



Figure 3: Microscopic images (top), and the probability distributions of the positional errors of wires from nominal positions (bottom) of measured wire-grids.



Figure 4: Transmission and reflection coefficients of regular ($\sigma_{\eta} = 0$) and irregular wire-grids for TE wave incidence. ($2a = 10\mu$ m, $h = 25\mu$ m, $\chi = 45^{\circ}$.) E_{perp} and E_{para} denote the cases that the incident electric field is perpendicular and parallel to the plane of incidence, respectively. G_{perp} and G_{para} denote the cases that the wire is perpendicular ($\phi_g = 0^{\circ}$) and parallel ($\phi_g = 90^{\circ}$) to the plane of incidence, respectively.



Figure 5: Transmission and reflection coefficients of regular ($\sigma_{\eta} = 0$) and irregular wire-grids for TM wave incidence. Same as in Fig. 4.

respectively. Measurement results for Grid #1 and Grid #2 are also shown in these Figures. The corresponding results of calculations and measurements for the case of TM-wave incidence are also shown in Figs. 5(a) and (b).

By comparing Fig. 4 and Fig. 5, it is found that the transmission and reflection characteristics of wire-grid are significantly affected by the grid irregularity when the incident wave has electric field component parallel to the grid wires (TM-wave incidence) as shown in Fig. 5. while the grid irregularity has little effect on its characteristics when the electric field of the incident wave was perpendicular to the grid wires (TE-wave incidence) as shown in Fig. 4. This can be explained by the fact that large currents are effectively induced on grid wires in the TM-wave incident case where the incident wave has a electric field component parallel to the wires, while currents are not effectively induced on grid wires in the TE-wave incident case where the incident wave does not have electric field component parallel to the wires.

Measurement results for Grid #1 and Grid #2 are also shown in Figs. 4(b) and 5(a) for comparison. From comparison between these measurement and calculation results, it is found that the measurement results for Grid #1 whose standard deviation of wire position irregularity σ_{η} was 22.7 % of the nominal wire spacing h agrees well with the theoretically calculated characteristics for $\sigma_{\eta} = 0.20h$. For the more irregular Grid #2

whose standard deviation of wire position irregularity was 52 % of h, the general trend of the different dependence of the characteristics on the grid irregularity between the TM and TE-wave incident cases agrees well with the measurement results. although the theoretical calculation results are found to overestimate the effects of irregularity as compared with the measurement results. This suggests that the irregularity of 52 % of wire spacing might be too large for the perturbation theory of irregular grid presented in Section 5 to be applicable.

7 Conclusion

In the first half of this paper, an accurate and computationally efficient method was presented for calculating the transmission and reflection characteristics of wire-grids of a periodic array of cylinders for arbitrary angles of incidence and grid rotation.

In the latter half of this paper, a method to calculate the effects of slight irregularity in wire spacing of grid was proposed by treating the irregular displacements of wire positions as a small perturbation to the theory of regular grid derived in the first half of this paper. Measurements made for actual wire-grids in millimeter- and submillimeter wavelengths have demonstrated that this perturbation theory was applicable to slightly irregular wire-grids whose standard deviation of the displacements of wire positions is less than about 30 % of the nominal wire spacing.

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