

PHYSICS 831 PLAN

=====

1. TOPICS:

A. Review of basic relations on fields, potentials, sources

(Revision of undergraduate material and familiarity with notation)

B. Radiation from single charges

Lienard-Wiechert potentials, fields of moving charges, velocity fields, ionization by fast particles, virtual photons, radiation from accelerated particles at low velocities, Larmor's Formula, Thomson cross-section, damping and radiation reaction, forced oscillators, Rayleigh scattering, directivity and polarization of Thomson scattering, radiation at large velocities (Lienard's Formula).

C. Radiation from ensembles

Multipole expansion, dipole approximation, magnetic dipole and electric quadrupole radiation, radiation from identical particles.

D. Radiation from antennas

Formulation of antenna problems, Schelkunoff Vector, linear antennas, radiation resistance, measures of directivity, loop antenna

E. Scalar diffraction theory

Fields as sources of radiation, Kirchhoff formulation, Fresnel and Fraunhofer diffraction, angular and spatial spectra of radiation, array theory, applications of Fourier Transform relationships in image processing, vector theory (brief).

F. Propagation in material media

Free electron theory, plasma effects in metals and gases, anisotropic media, magnetoionic theory and Faraday effect.

G. Other topics if time permits (usually doesn't).

2. BOOKS:

No book exactly parallels the course. Those using electromagnetics in their research, or taking 832* later, may wish to purchase Jackson, "Classical Electrodynamics" (Wiley). It's a good reference book but not so great to learn the subject from. On Reserve for 831* in the Physics Library are:

Jackson - Classical Electrodynamics

Marion - Classical Electromagnetic Radiation

Panofsky and Phillips - Classical Electricity and Magnetism

Papas - Theory of Electromagnetic Wave Propagation

Ramo, Whinnery and van Duzer - Fields and Waves in Communications Electronics

These will all be referred to from time to time in class. The last is an undergraduate engineering book, strong on antennas, etc. You may well be able to get by without a text by using these Reserve books in the Library.

A.H.Bridle Sept 82

Units in Electromagnetism (see Appendix in Jackson)

We introduce arbitrary constants in two force laws:

Electrostatic force between charges $\underline{F}_e = \frac{k_e Q_1 Q_2 \underline{R}}{R^3}$

Magnetostatic force between currents $\underline{F}_m = k_m I_1 I_2 \oint \oint \frac{d\underline{L}_1 \times (d\underline{L}_2 \times \underline{R})}{R^3}$

Dimensionally $[F] = [k_e] \frac{[Q^2]}{[L^2]} = [k_m] \frac{[Q^2]}{[T^2]}$

$$\frac{[k_e]}{[k_m]} \underline{\text{fixed}} = \frac{[L]^2}{[T]^2}, \quad \underline{\text{observe}} = c^2$$

RMKS Write $k_e = \frac{1}{4\pi\epsilon_0}$ $k_m = \frac{\mu_0}{4\pi}$

Suppresses 4π factors in Maxwell's Eqs.

Keep engineering unit of current (Ampere)

$$\mu_0 = 4\pi \times 10^{-7} \text{ nt/A}^2$$

$$\rightarrow \epsilon_0 = \frac{10^7}{4\pi c^2} \text{ farad/m}$$

Disadvantage $\rightarrow \underline{E}$ and $\underline{B}$ different dimensions (Volts/m, Volt-sec/m²)

Gaussian Write $k_e = 1$, $k_m = 1/c^2$ (Choose $[Q] = [ML^{3/2}T^{-2}]$)

Use c.g.s. units throughout

Merit $\rightarrow \underline{E}$ and $\underline{B}$ same dimensions

$$\underline{F} = q \left(\underline{E} + \frac{\underline{v}}{c} \times \underline{B} \right) \quad \text{looks like a Lorentz-transformation term.}$$

Jackson gives tables for RMKS / Gaussian conversion

Most practical applications use RMKS }
Theoretical texts use Gaussian }

Classical Nonrelativistic Development of Field Equations

Static

1) Fields from the Force Laws. (Free Space only, at first)

Historically, it was found that bodies could be "electrified" by friction, heat, chemical action, etc. so that static electrified bodies exert forces on one another at a distance. Then found from experiments in air that could assign attribute of CHARGE, q , to each "electrified" body so that forces exerted on and by that body are both proportional to q .

Also find inverse square law

central force

linear superposition

This situation summed up in Coulomb's Law:

$$\underline{F}_i = q_i \sum_j \frac{q_j \underline{R}_{ij}}{4\pi\epsilon_0 R_{ij}^3} \quad \underline{R}_{ij} = \underline{r}_i - \underline{r}_j$$

Hence define $\underline{E}(\underline{r}_i) = \lim_{q_i \rightarrow 0} \left(\frac{\underline{F}_i}{q_i} \right)$ to be continuous vector field.

$$\underline{E}(\underline{r}_i) = \lim_{q_i \rightarrow 0} \sum_j \frac{q_j \underline{R}_{ij}}{4\pi\epsilon_0 R_{ij}^3}$$

This seems straightforward enough until you recognise that the intermediate step really involves considering a test body, volume V , charge density ρ , momentum $\underline{p} = m\underline{v}$ with $\underline{v}$ small, for which

$$\underline{E} = \lim_{\substack{v \rightarrow 0 \\ \tau \rightarrow 0 \\ \rho V \rightarrow 0 \\ V \rightarrow 0}} \left\{ \frac{\underline{p}(t+\tau) - \underline{p}(t)}{\rho V \tau} \right\}$$

$v \rightarrow 0$ is necessary to exclude "magnetic effects", $\tau \rightarrow 0$ and $V \rightarrow 0$ are necessary for infinity, and $\rho V \rightarrow 0$ is necessary to avoid self-force trouble

3

In fact we know that we cannot take limit $V \rightarrow 0$ and still measure p , also we cannot take limit $pV \rightarrow 0$ because of charge quantisation. This is what makes the theory "classical" — at its very basis it ignores both quantisation and the uncertainty principle.

Normally when developing the theory we shrug this off and say that we will not consider the theory to represent the actual microscopic level of electrical phenomena, but to represent a "suitable" spatial and temporal average over many microscopic systems. These averages are supposed to be taken over macroscopically infinitesimal volumes and durations which nevertheless smooth out real quantum discreteness and fluctuations. Yet time and again we will return to discussion of individual particles and use of "macroscopic" electromagnetic concepts in descriptions of microscopic systems. It is not obvious that the classical theory should be so robust, and the physicist should not take its seeming robustness for granted.

For example, the problems of a classical finite electron are nontrivial:

How does the E-field of a finite charge act on itself?

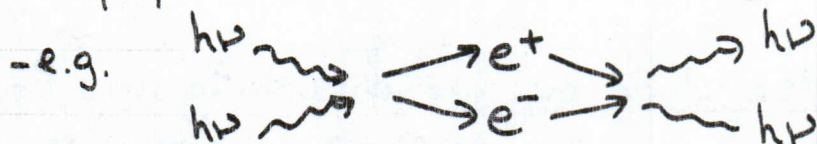
Is there microscopic saturation of the E-field to limit self-energies of real particles?

Does linear superposition hold at all ranges and all field strengths?

— for a thorough discussion of the classical problems associated with descriptions of particles, see ROHRICH, "Classical Charged Particles".

If we add obvious quantum effects, things get rapidly worse

- uncertainties in localisation, charge distribution
- can a localised electron be nearly at rest, so that electric and magnetic effects are properly separable?
- is superposition safe to assume at high energies



emergent wave fields are not necessarily linear superposition of entrant wave fields.

Normally say we will avoid all of this by restricting ourselves to

- 1) describing phenomena on scales $\gg$ de Broglie wavelengths of particles
- 2) restricting frequencies so that $h\nu \ll m_0 c^2$ for particles
- 3) phenomena involving such numbers of particles that we can effectively treat charge as a continuous fluid whose charge density

$$\rho = \sum_V q_i \delta(\underline{r} - \underline{r}_i) \quad (\underline{r} \text{ within } V)$$

is derived from a finite but small volume average around $\underline{r}$.

Mobility of the charged fluid then sets up a current density

$$\underline{J} = \sum_V q_i \underline{v}_i \delta(\underline{r} - \underline{r}_i)$$

and conservation of charge requires

$$\nabla \cdot \underline{J} + \dot{\rho} = 0$$

(We will return to the constraints on the averaging process later)

Then it is best to assert that our experimental experience with charged-particle assemblies shows that the general force density experienced by a moving charge distribution that does not radiate significantly is, in the rest-frame of the laboratory observer,

$$\underline{\underline{f}} = \rho \underline{\underline{E}} + \underline{\underline{J}} \times \underline{\underline{B}}$$

as if the force on an individual charged particle were

$$\underline{\underline{F}} = q(\underline{\underline{E}} + \underline{\underline{v}} \times \underline{\underline{B}})$$
 even though we can't really show this is true.

- i.e. we assert that
- a) charge does not appear to depend on velocity (if it did, atoms could not preserve overall electrical neutrality, as observed, and "neutral" atoms or molecules could be deflected)
 - b) the total force has a velocity-independent part that we associate with $\underline{\underline{E}}$ and a velocity-proportional part that we associate with $\underline{\underline{B}}$.

Assertion (b) is really a result of failure to build a properly relativistic theory right from the start, and to describe all phenomena in the rest-frame of the observer. In a properly relativistic theory, the only forces on charges are electrostatic forces in the charges' own rest-frames. The velocity-dependent component comes from the transformation of the force from the particle's rest-frame to the observer's frame.

We then appeal to experiment again to suggest how to relate $\underline{\underline{E}}$ and $\underline{\underline{B}}$ to their sources.

With the "continuous-fluid" averaging procedure, our experience of Coulomb's law from static charge distributions becomes summarised as:

$$\underline{E}(\underline{r}_f) = \iiint_V \frac{\rho(\underline{r}_s) \underline{R}}{4\pi\epsilon_0 R^3} dV, \quad \underline{R} = \underline{r}_f - \underline{r}_s$$

where $\underline{r}_f$ is vector coordinate of a field point at which $\underline{E}$ is to be evaluated

and $\underline{r}_s$ is vector coordinate of a source point at which ρ is known

and V is a volume including all $\underline{r}_s$ for which $\rho(\underline{r}_s) \neq 0$.

The analogous statement about magnetic forces comes experimentally from the Biot-Savart law, which is demonstrated from observations with steady currents in complete circuits:

$$\underline{B}(\underline{r}_f) = \mu_0 \iiint_V \frac{\underline{J}(\underline{r}_s) \times \underline{R}}{4\pi R^3} dV$$

It is an assumption of classical nonrelativistic electromagnetism that this expression holds for current elements (incomplete current circuits) ^(later) and/or accelerated charges. The need for this assumption disappears in a fully relativistic treatment (e.g. ROSSER, "Classical Electromagnetism via Relativity").

The Biot-Savart law relates $\underline{B}$ to its $\underline{J}$ sources analogously to the relation of $\underline{E}$ to its ρ sources through Coulomb's Law. Note the different symmetries of the fields:

$\underline{E}, \underline{J}$ are changed in sign if we invert the co-ordinate system

$\underline{B}, \rho$ are unchanged

$\underline{B}, \underline{J}$ are changed in sign under time reversal

$\underline{E}, \rho$ are unchanged

The Biot-Savart Law also leads us to the form of the magnetic (vector) potential, as the following argument shows:

Noting that $\nabla_f \left(\frac{1}{R} \right) = -\frac{\underline{R}}{R^3}$, we can write the Biot-Savart Law

$$\begin{aligned}\underline{B}(\underline{r}_f) &= \frac{\mu_0}{4\pi} \iiint \underline{J}(\underline{r}_s) \times -\nabla_f \left(\frac{1}{R} \right) dV \\ &= \frac{\mu_0}{4\pi} \iiint \nabla_f \left(\frac{1}{R} \right) \times \underline{J}(\underline{r}_s) dV\end{aligned}$$

Now use vector identity $\nabla \times (\psi \underline{u}) = (\nabla \psi) \times \underline{u} + \psi (\nabla \times \underline{u})$

i.e. $(\nabla \psi) \times \underline{u} = \nabla \times (\psi \underline{u}) - \psi (\nabla \times \underline{u})$

$$\begin{aligned}\text{Then } \nabla_f \left(\frac{1}{R} \right) \times \underline{J}(\underline{r}_s) &= \nabla_f \times \left(\frac{\underline{J}(\underline{r}_s)}{R} \right) - \frac{1}{R} \underbrace{\nabla_f \times \underline{J}(\underline{r}_s)}_{=0 \text{ identically}} \\ &= \nabla_f \times \left(\frac{\underline{J}(\underline{r}_s)}{R} \right)\end{aligned}$$

$$\text{so } \underline{B}(\underline{r}_f) = \frac{\mu_0}{4\pi} \iiint \nabla_f \times \left(\frac{\underline{J}(\underline{r}_s)}{R} \right) dV$$

$$= \nabla_f \times \frac{\mu_0}{4\pi} \iiint \frac{\underline{J}(\underline{r}_s)}{R} dV$$

because $\nabla_f \times$
and $\iiint$ over $\underline{r}_s$
commute

$$\text{i.e. } \underline{B}(\underline{r}_f) = \nabla_f \times \underline{A}(\underline{r}_f)$$

where $\underline{A}(\underline{r}_f)$ is a vector potential related to sources $\underline{J}(\underline{r}_s)$ via

$$\underline{A}(\underline{r}_f) = \frac{\mu_0}{4\pi} \iiint \frac{\underline{J}(\underline{r}_s)}{R} dV$$

These relations are vector analogues of :-

$$\begin{aligned}\underline{E}(\underline{r}_f) &= -\nabla_f \phi(\underline{r}_f) \\ \phi(\underline{r}_f) &= \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\underline{r}_s)}{R} dV\end{aligned}$$

From these we can derive differential relationships for the static (time-stationary) fields arising from steady currents and static charges:

$$\underline{E} = -\underline{\nabla}\phi \text{ implies } \underline{\nabla} \times \underline{E} = -\underline{\nabla} \times \underline{\nabla}\phi \equiv 0 \text{ identically}$$

$$\underline{B} = \underline{\nabla} \times \underline{A} \quad \dots \quad \underline{\nabla} \cdot \underline{B} = \underline{\nabla} \cdot (\underline{\nabla} \times \underline{A}) \equiv 0 \quad \dots$$

$$\underline{E} = \iiint_V \frac{\rho(\underline{r}_s) \underline{R}}{4\pi\epsilon_0 R^3} dV \text{ implies } \underline{\nabla} \cdot \underline{E} = \rho/\epsilon_0$$

$$\underline{B} = \iiint_V \frac{\mu_0 \underline{J}(\underline{r}_s) \times \underline{R}}{4\pi\epsilon_0 R^3} dV \rightarrow \underline{\nabla} \times \underline{B} = \mu_0 \underline{J} \quad (\text{provided } \underline{\nabla} \cdot \underline{J} = 0)$$

Note the symmetry:

	$\underline{\nabla} \cdot$	$\underline{\nabla} \times$	
$\underline{E}$	I	H	(I - inhomogeneous H - homogeneous)
$\underline{B}$	H	I	

The validity of the homogeneous (H) equations amounts to the statement that the only known sources of macroscopic $\underline{B}$ -fields are current loops, i.e. there are no macroscopic free magnetic monopoles (and hence no monopole currents to give static-field $\underline{E}$ -configurations with nonvanishing curl).

2) Time-Variable Fields

The generalisation from these static-field equations to general equations involving time-variable fields and accelerated charges is on a much less secure experimental basis in a nonrelativistic theory.

- ① We assume that the $\underline{\nabla}$ equations apply equally well to time-variable fields associated with variable currents in incomplete circuits, and thus with accelerated charges.

i.e. we assert that $\nabla \cdot \underline{E} = \rho/\epsilon_0$ and $\nabla \cdot \underline{B} = 0$ are always true in free space.

The modification to the $\nabla \times \underline{E}$ equation is usually justified by appeal to experiments involving the appearance of emfs in circuits in which there are changes in flux linkages — e.g. in moving circuits (the dynamo effect) or in fixed circuits threaded by variable magnetic flux (the transformer effect). It is important to realise that Maxwell's $\nabla \times \underline{E}$ equation goes well beyond these experiments and asserts that there is an underlying VACUUM effect that has nothing to do with the media involved in the dynamo or transformer phenomena —

② Assert that: $\boxed{\nabla \times \underline{E} + \dot{\underline{B}} = 0}$, noting that this

is a free-space equation whose implications are only directly verified by experiment in situations such as the betatron accelerator. In this sense it may be misleading still to call it "Faraday's Law of Induction." It is invoked for example to explain the production of low-energy cosmic-ray particles as a result of the collapse of magnetised interstellar gas clouds under their own gravity — possibly the closest to an all-vacuum example of its applicability. Again, a properly relativistic theory makes it clear why this generalisation is correct, and the evidence for relativity's correctness is indirect evidence for the validity of this law.

③ Note that $\nabla \times \underline{B} = \mu_0 \underline{J}$ cannot be generally true because $\nabla \cdot (\nabla \times \underline{B}) \equiv 0$
but $\nabla \cdot \underline{J}$ can $\neq 0$

This is rescued via the continuity eqn. $\nabla \cdot \underline{J} + \dot{\rho} = 0$

10
Use "intuition" to equate two identical nulls $\nabla \cdot (\nabla \times \underline{B}) = (\nabla \cdot \underline{J} + \dot{\rho}) \mu_0$

Now write $\dot{\rho} = \epsilon_0 \nabla \cdot \dot{\underline{E}} \rightarrow \nabla \cdot (\nabla \times \underline{B}) = \nabla \cdot \mu_0 (\underline{J} + \epsilon_0 \dot{\underline{E}})$

Then assert that quantities inside the divergence are generally equal:

$$\nabla \times \underline{B} = \mu_0 (\underline{J} + \epsilon_0 \dot{\underline{E}})$$

Note that this is a "free-space" equation, nothing implicitly to do with material media such as dielectrics, etc. which are usually invoked in discussions of "displacement current".

It is actually misleading to refer to the "new" term in this equation as a "current" because it happens to behave mathematically like $\underline{J}$. $\epsilon_0 \dot{\underline{E}}$ is NOT a current which produces a $\underline{B}$ -field via the Biot-Savart law, nor is it a current on which there can be a force via the Lorentz force law. Better to call $\epsilon_0 \dot{\underline{E}}$ the "Maxwell term".

What this equation really says is that REAL charge motions that produce NON-STEADY currents for which $\nabla \cdot \underline{J} \neq 0$ produce $\underline{B}$ -fields different from those of STEADY currents. These same charge motions are associated with time-variations of $\underline{E}$, so that locally and instantaneously:—

$$\nabla \times \underline{B} - \mu_0 \epsilon_0 \dot{\underline{E}} = \mu_0 \underline{J}$$

There is an excellent discussion of Maxwell's Equations and their logical development in ROSSER, "Classical Electromagnetism Via Relativity", which also shows how the entire theory can be obtained from Coulomb's law and the Principles of Special Relativity.

11

Differential

3) Equations Relating Potentials To their General Sources

The ad hoc aspect of the new term is clear from its derivation in the purely classical, nonrelativistic theory. Its justification is usually its "prediction" of radiation phenomena, with a unique free-space velocity $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$.

This is usually treated via the inhomogeneous equations for the potentials: (e.g. Jackson, Ch. 6)

$$\underline{\nabla} \times \underline{E} + \underline{\dot{B}} = 0 \Rightarrow \underline{\nabla} \times (\underline{E} + \underline{\dot{A}}) = 0, \text{ i.e. } \underline{E} + \underline{\dot{A}} = \underline{\nabla} f \text{ (f = some scalar)}$$

To get correct static-field limit, we put $f = -\phi$, $\rightarrow \underline{E} = -\underline{\nabla} \phi - \underline{\dot{A}}$ when situation is other than time-stationary.

$$\begin{aligned} \text{Then take } \underline{\nabla} \times \underline{B} - \mu_0 \epsilon_0 \underline{\dot{E}} &= \mu_0 \underline{J} \\ \downarrow \\ \underline{\nabla} \times \underline{\nabla} \times \underline{A} - \mu_0 \epsilon_0 \underline{\dot{E}} &= \mu_0 \underline{J} \\ \downarrow \\ \underline{\nabla} (\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A} - \mu_0 \epsilon_0 \underline{\dot{E}} &= \mu_0 \underline{J} \\ \downarrow \\ \underline{\nabla} (\underline{\nabla} \cdot \underline{A}) - \nabla^2 \underline{A} + \mu_0 \epsilon_0 \underline{\nabla} \phi + \mu_0 \epsilon_0 \underline{\ddot{A}} &= \mu_0 \underline{J} \end{aligned}$$

$$\text{i.e. } \nabla^2 \underline{A} - \mu_0 \epsilon_0 \underline{\ddot{A}} = -\mu_0 \underline{J} + \underline{\nabla} (\underline{\nabla} \cdot \underline{A} + \mu_0 \epsilon_0 \phi)$$

In this nonrelativistic classical theory we arbitrarily choose the gauge of the potentials so that $\underline{\nabla} \cdot \underline{A} + \mu_0 \epsilon_0 \phi = 0$, reducing this mixed $(\underline{A}, \phi)$ equation to the simple inhomogeneous wave equation between $\underline{A}$ and its source $\underline{J}$:

$$\nabla^2 \underline{A} - \mu_0 \epsilon_0 \underline{\ddot{A}} = -\mu_0 \underline{J}$$

The same choice of gauge for the potentials also leads to

$$\nabla^2 \phi - \mu_0 \epsilon_0 \ddot{\phi} = -\rho/\epsilon_0$$

In the non-relativistic theory this simplification (separation) of the wave equations for $\underline{A}$ and ϕ is the main justification for use of the Lorentz gauge $\underline{\nabla} \cdot \underline{A} + \mu_0 \epsilon_0 \dot{\phi} = 0$. In the relativistic theory this choice appears naturally as the only Lorentz-invariant choice.

It is worth recalling why we have freedom of choice of gauge at all.

Suppose a vector potential $\underline{A}_1(\underline{r}_f)$ satisfactorily gives a certain magnetic field distribution $\underline{B}(\underline{r}_f)$.

$$\text{i.e. } \underline{B} = \underline{\nabla} \times \underline{A}_1 \text{ indeed.}$$

Now invent $\underline{A}_2 = \underline{A}_1 + \underline{\nabla} \psi$ where ψ is an arbitrary scalar fn. of $(\underline{r}, t)$

$$\text{Then } \underline{\nabla} \times \underline{A}_2 = \underline{\nabla} \times \underline{A}_1 + \underline{\nabla} \times \underline{\nabla} \psi \rightarrow 0 \text{ whatever the form of } \psi(\underline{r}, t)$$

i.e. $\underline{A}_2$ also satisfactorily $\rightarrow$ SAME $\underline{B}(\underline{r}_f)$. This parallels the scalar-constant arbitrariness of the scalar potential ϕ .

The "equivalent" vector potentials are distinguishable by their divergences: $\underline{\nabla} \cdot \underline{A}_2 = \underline{\nabla} \cdot \underline{A}_1 + \nabla^2 \psi$

So we must impose a condition on $\underline{\nabla} \cdot \underline{A}$ to make $\underline{A}$ unique.

The condition imposed is called the GAUGE.

As it's choice does not affect $\underline{B}$, we say $\underline{B}$ is GAUGE INVARIANT
(Note that if $\phi \rightarrow (\phi - \partial \psi / \partial t)$, $\underline{E}$ is not changed either)
Simultaneously

4) Solutions To the Inhomogeneous Wave Equation

The empirical justification for all this arbitrariness, especially of the "Maxwell term", in non-relativistic theory, is the form of the solutions to the inhomogeneous equations for the potentials:

- see Jackson, Sec. 6.6; Panofsky & Phillips, Ch. 14

The inhomogeneous equations have the general form:

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \Psi(\underline{r}, t) = -f(\underline{r}, t)$$

- here $\underline{r}$ is a general spatial co-ordinate, whether a source is there or not

- where the potential Ψ stands for A_x, A_y, A_z or ϕ
 - where the source f $\mu_0 J_x, \mu_0 J_y, \mu_0 J_z$ or ρ/ϵ_0

Any physically realisable source function $f(\underline{r}, t)$ will be Fourier analysable:

i.e. $f(\underline{r}, t) = \int_{-\infty}^{+\infty} F(\underline{r}, \omega) e^{-i\omega t} d\omega$, where

$$F(\underline{r}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\underline{r}, t) e^{i\omega t} dt \text{ is the frequency spectrum of } f(\underline{r}, t)$$

Similarly, any realisable potential $\Psi(\underline{r}, t)$ will be analysable into a frequency spectrum $\Psi(\underline{r}, \omega)$.

Substitute the Fourier representations of Ψ and f into the inhom. eqn:

$$(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}) \int_{-\infty}^{\infty} \Psi(\underline{r}, \omega) e^{-i\omega t} d\omega = - \int_{-\infty}^{\infty} F(\underline{r}, \omega) e^{-i\omega t} d\omega$$

i.e. $\int_{-\infty}^{\infty} (\nabla^2 + \frac{\omega^2}{c^2}) \Psi(\underline{r}, \omega) e^{-i\omega t} d\omega = - \int_{-\infty}^{\infty} F(\underline{r}, \omega) e^{-i\omega t} d\omega$

This integral relation can be satisfied only if:

$$(\nabla^2 + \frac{\omega^2}{c^2}) \Psi(\underline{r}, \omega) = -F(\underline{r}, \omega)$$

at each and every ω

$$(\nabla^2 + k^2) \Psi(\underline{r}, \omega) = -F(\underline{r}, \omega)$$

NOTATION
 $k = 2\pi/\lambda = \omega/c$

At a given frequency ω , Ψ takes a spatial form $\Psi_\omega(\underline{r})$. This ω -freq. component of the potential satisfies a purely SPATIAL equation relating it to the ω -freq. component $F_\omega(\underline{r})$ of the source function

$$(\nabla^2 + k^2) \Psi_\omega(\underline{r}) = -F_\omega(\underline{r})$$

Now we recognise that linearsuperposition means that we can CONSTRUCT Ψ_ω for any given F_ω by superposing the potentials $\Psi_\omega(\underline{r}, \underline{r}_s)$ produced by unit point sources at locations $\underline{r}_s$

$$\text{i.e. if } (\nabla^2 + k^2) \Psi_\omega(\underline{r}, \underline{r}_s) = -\delta(\underline{r} - \underline{r}_s) \quad (a)$$

$$\text{and } F_\omega(\underline{r}) = \iiint_{\text{all sources}} F_\omega(\underline{r}_s) \delta(\underline{r} - \underline{r}_s) dV_s$$

$$\underline{\text{then}} \quad \Psi_\omega(\underline{r}) = \iiint_{\text{all sources}} F_\omega(\underline{r}_s) \Psi_\omega(\underline{r}, \underline{r}_s) dV_s \quad (b)$$

So the general problem reduces (a) to solving the δ -fn equation (a)
(b) integral (b)

In math. phys. language the $\Psi_\omega(\underline{r}, \underline{r}_s)$ will be the GREEN'S FUNCTIONS $G(\underline{r}, \underline{r}_s)$ for the problem.

Now to find the $\Psi_\omega(\underline{r}, \underline{r}_s)$ forms:-

These solutions are well-known functions.

The equation to be solved is $(\nabla^2 + k^2) \Psi = 0$

everywhere except $\underline{r} = \underline{r}_s$.

The required solution is obviously spherically symmetric in the co-ordinate $\underline{R} = \underline{r} - \underline{r}_s$, in which it is a solution of the total d.e.:

$$\frac{1}{R} \frac{d^2}{dR^2} (R\Psi) + k^2 \Psi = 0$$

This has solutions in the general form:

$$\Psi(R) = \frac{A}{R} e^{\pm ikR}$$

To evaluate the constant A consider the volume integral of the d.e. over a small neighbourhood around $\underline{r} \sim \underline{r}_s$, where $R \rightarrow 0$ and $\Psi \rightarrow A/R$

$$\iiint_{\substack{\underline{r} \rightarrow \underline{r}_s \\ \text{i.e. } R \rightarrow 0}} (\nabla^2 + k^2) \frac{A}{R} dV \text{ must} = - \iiint_{\underline{r} \rightarrow \underline{r}_s} \delta(\underline{r} - \underline{r}_s) dV$$

first term

$$\begin{aligned} \iiint \nabla^2 \left(\frac{A}{R} \right) dV &= A \iiint \underline{\nabla} \cdot \underline{\nabla} \left(\frac{1}{R} \right) dV \\ &= A \oint \underline{\nabla} \left(\frac{1}{R} \right) \cdot d\underline{S} \\ &= A \oint \frac{-\underline{R}}{R^3} \cdot d\underline{S} \\ &= A \oint -d\Omega \\ &= -4\pi A \end{aligned}$$

Second term

$$\begin{aligned}
 \iiint k^2 \frac{A}{R} dV &= k^2 A \iiint \frac{1}{R} \cdot R^2 dR d\Omega \\
 &= k^2 A \oint d\Omega \int R dR \\
 &\rightarrow 0 \text{ as } R \rightarrow 0
 \end{aligned}$$

Third term -1 by definition

$$\text{i.e.} \quad -4\pi A + 0 = -1$$

$$A = 1/4\pi$$

$$\Psi(R) = \frac{1}{4\pi R} e^{\pm i k R}$$

$$\text{i.e.} \quad \boxed{\Psi_w(\underline{r}, \underline{r}_s) = \frac{1}{4\pi |\underline{r} - \underline{r}_s|} e^{\pm i k |\underline{r} - \underline{r}_s|}}$$

Using these as the Greens functions we write the spectrum of the potential corresponding to the source spectrum $F(\underline{r}, \omega)$:

$$\Psi(\underline{r}, \omega) = \iiint_{\substack{\text{all} \\ \text{Sources}}} \frac{F(\underline{r}_s, \omega)}{4\pi R} e^{\pm i k R} dV_s$$

With this as the spectrum, the Fourier transform gives for the time-varying potential:

$$\psi(\underline{r}, t) = \iiint_{\substack{\text{all} \\ \text{Sources}}} \int_{-\infty}^{+\infty} \frac{F(\underline{r}_s, \omega)}{4\pi R} e^{-i(\omega t \mp k R)} dV_s d\omega$$

Now write the exponential $\omega t \mp k R = \omega(t \mp \frac{R}{c}) = \omega t \mp$
 (THIS DEFINES NOTATION $t_{\mp}$)

With the $t_{\pm}$ notation,

$$\psi(\underline{r}, t) = \iiint_{\text{all sources}} \frac{1}{4\pi R} \left\{ \int_{-\infty}^{+\infty} F(\underline{r}_s, \omega) e^{-i\omega t_{\pm}} d\omega \right\} dV_s$$

But the quantity in the inner $\{ \}$ brackets is the Fourier representation of $f(\underline{r}, t)$ with $\underline{r}_s$ for $\underline{r}$ and $t_{\pm}$ for t

i.e. $\psi(\underline{r}, t) = \iiint_{\text{all sources}} \frac{f(\underline{r}_s, t_{\pm})}{4\pi R} dV_s$ is the general solⁿ.

Explicitly in terms of the potentials and sources, for a particular field point $\underline{r} = \underline{r}_f$

$$\phi(\underline{r}_f, t) = \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\underline{r}_s, t_{\pm})}{R} dV_s$$

$$\underline{A}(\underline{r}_f, t) = \frac{\mu_0}{4\pi} \iiint \frac{\underline{J}(\underline{r}_s, t_{\pm})}{R} dV_s$$

where $t_{\pm} = t \pm R/c$

$$R = \underline{r}_f - \underline{r}_s$$

Note how field point co-ordinates $\underline{r}_f$ enter into the integrand:-

- 1) in normalising source contributions inversely as the distance from the sources to the field point
- 2) in evaluating the time $t_{\pm}$ at which any given source's contribution to the potential at $(\underline{r}_f, t)$ is entered.

These results embody the notion that "news" about the behaviour of ρ and $\underline{J}$ propagates into the form of the potential at distant points at a finite speed $c = 1/\sqrt{\epsilon_0\mu_0}$ in vacuum, and the agreement between this and experimental experience ultimately justifies the "Maxwell term".

The expressions with the t_- terms require that when we compute a potential at $(\underline{r}_f, t)$ we sum the static (Coulomb, Biot-Savart) potentials of the sources as they were at the appropriate EARLIER times $(t - R/c)$.

These RETARDED POTENTIALS are therefore sums of the static potential contributions from the sources as they "were" at earlier times, as if the "news" of each contribution to the potential travels forwards in time at velocity c .

The expressions with the t_+ terms represent contributions from sources as they will be at LATER times $(t + R/c)$. These ADVANCED POTENTIALS correspond to contributions from source configurations that have not yet occurred by the time we evaluate $\phi(\underline{r}_f, t)$, $\underline{A}(\underline{r}_f, t)$. Including these terms would allow that effects (contributions to the potentials) could PRECEDE their causes (source configurations) in time sequence.

Our basic theory is completely time-symmetric in this sense, but our experience of COMPLEX systems is that a unique time-sense (ARROW of time) exists in which entropy increases and effects follow causes. We therefore IMPOSE a time-arrow on the electromagnetic theory to correspond to this statistical sense of CAUSALITY, and DISCARD ADVANCED POTENTIALS

A useful aid to visualising the computation of the retarded potentials is to imagine an "information-gathering sphere" collapsing onto the field point at time t . The surface of this sphere is a light pulse that "illuminates" sources at times $t - R/c$ when the sphere radius is R . The potentials at $(\underline{r}_f, t)$ are then the Coulomb and Biot-Savart potentials of all the sources illuminated by the collapsing light-sphere, at their "illuminated" distances $R = \underline{r}_f - \underline{r}_s(t_-)$

5) General Nature of Problems in Electromagnetism

Before proceeding to apply the very powerful integral relations we have just developed for the potentials in terms of the sources at RETARDED times, it is worth making an overview of the nature of problems in electromagnetism and electrodynamics.

Clearly, the most general problem involves interacting fields and sources. We envisage specifying a set of initial fields and sources, i.e. charges in motion. In principle, using the Lorentz force law we can say how the source distribution changes with time under the action of the fields, and how the fields are modified by changes in the source distribution. We can then predict all future field and source distributions from the initial conditions. In practice, all but the simplest cases are mathematically appalling if fields and sources strongly interact and modify one another.

The tractable problems boil down to :

1) Problems in which fields and sources are constant in time — electrostatics and magnetostatics. Here all retardation effects disappear and problems reduce to solutions of Poisson's Equation $\nabla^2 \psi = f$ with appropriate boundary conditions

2) Problems in which charged particles move in constant or slowly-varying prescribed fields — particle electrodynamics. We may be interested in the particle trajectories, or in the radiation which they produce as a result of their trajectories, but the particles do not control the fields defining the underlying problem. — e.g. cosmic rays in geomagn. field motion of particles in accelerators, electron microscopes, or the Van Allen belts.

3) Problems in which we study the fields produced by prescribed source configurations, with only secondary attention to the fields producing those configurations. Here we ^{also} assume that the fields produced by the sources do not modify the fields controlling the source configuration. The classic examples of this class of problem are problems of radiation from charged particles or from antennas.

4) Problems in which we analyse the propagation of radiation fields, which are the class of field capable of detaching from their sources and then leading an essentially independent existence of their own. Propagation problems may involve interactions with ambient fields and sources (material media). Examples are problems in refraction, reflection and scattering of radiation in dielectrics, metals, plasmas, etc. In these problems we ignore the sources of the radiation fields, i.e. we assume they are not reacted on by any changes in the radiation fields.

Fortunately, a very wide range of applications to useful physics falls in these four categories, and the general fully-interacting situation need rarely be analysed in the laboratory. Exceptions are situations in plasma physics, magnetospheric physics and radio astrophysics, where we also encounter the complexities of relativistic charge motions in many applications.

In this course we will be concerned primarily with problems of types (3) and (4). We will begin by studying radiation from prescribed sources, initially examining the most elementary source of all, a single charge in general motion.

Fields due to Prescribed Sources

6) The Potentials and Fields of a Single Charge in General Motion

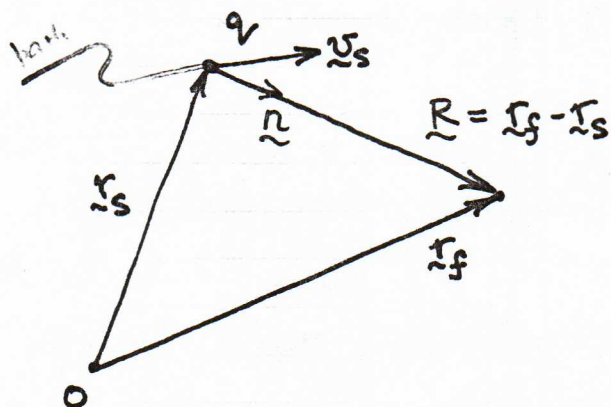
- see Jackson, Ch. 14 / Marion, Ch. 7 / (relativistically L/L "CTF" Ch. 8
Percosky & Phillips, Ch 19/20.

Consider a small but finite moving charge with position $\underline{r}_s(t)$, velocity $\underline{v}(t) = \dot{\underline{r}}_s(t)$. We ask for a formulation of its potentials and fields at a field point $\underline{r}_f$ at some particular time t . $\underline{r}_f$ is fixed, and the charge is "small" in the sense that we can neglect the variation of $\underline{R} = \underline{r}_f - \underline{r}_s$ across the charge.

The results for the potentials are a particularly useful form of the general solution to the inhomogeneous Helmholtz wave equation first derived by A. Liénard (1898) and E. Wiechert (1900), and known as the Liénard-Wiechert potentials. Their derivation is most easily visualised if we interpret the expression for the retarded potential as requiring a census of the charge "illuminated" by an "information gathering sphere" which collapses onto $\underline{r}_f$ at time t with velocity c . At any instant $t_- = t - R/c$ the sphere's radius is R and we count the total charge ρdV illuminated by the sphere at that moment and divide by $4\pi\epsilon_0 R$ to obtain an elementary contribution $d\phi$ (similarly counting current $\underline{J} dV$ and normalising by $\mu_0/4\pi R$ gives a contribution $d\underline{A}$).

With this procedure, there is a PURELY CLASSICAL effect whose result is that the NET CHARGE COUNTED IN A MOVING DISTRIBUTION CAN BE DIFFERENT FROM THE NET CHARGE COUNTED WHEN THE SAME DISTRIBUTION IS STATIC.

Start with the simplest case — a single moving point charge q



We want to calculate ϕ at a given field point $\underline{r}_f$ and a given value t of the time coordinate t' .

$\underline{r}_s$ is an arbitrary function $\underline{r}_s(t')$

$\underline{R}(t') = \underline{r}_f - \underline{r}_s(t')$ Vector from
Source
towards $\underline{r}_f$

$\underline{n}(t')$ unit vector $\parallel \underline{R}$ ($= \underline{R}/R$)

The integral we must evaluate for ϕ is:

$$\begin{aligned}\phi(\underline{r}_f, t) &= \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\delta(t' - t + \frac{|\underline{r}_f - \underline{r}_s(t')|}{c})}{|\underline{r}_f - \underline{r}_s(t')|} dt' \\ &= \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\delta(t' - t + \frac{R(t')}{c})}{R(t')} dt'\end{aligned}$$

The nontrivial aspect of this is that we are not integrating over the argument of the δ -function, because R is a function of t' .

Transform to the variable that is the δ -fn argument:

$$t'' = t' - t + \frac{R}{c}$$

$$\frac{dt''}{dt'} = 1 - 0 + \frac{1}{c} \frac{dR}{dt'}$$

$$= 1 + \frac{1}{c} \underline{\nabla}_s R \cdot \frac{d\underline{r}_s}{dt'}$$

$$= 1 + \underline{\nabla}_s R \cdot \underline{\beta}$$

$$(\underline{\beta} = \frac{\underline{v}_s}{c})$$

$$\text{Now } \underline{\nabla}_s R = -\underline{n},$$

$$\frac{dt''}{dt'} = 1 - \underline{n} \cdot \underline{\beta}$$

($\frac{dt}{dt'} = 0$ because t is fixed time)

$$\text{i.e. } \phi(\underline{r}_f, t) = \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\delta(t'')}{R(1-\beta \cdot \underline{n})} dt''$$

Now the integration gives us the value of the integrand at $t'' = 0$

i.e. at $t' = t - R/c$, the "retarded time"

$$\phi(\underline{r}_f, t) = \frac{q}{4\pi\epsilon_0 R(1-\beta \cdot \underline{n})} \Big|_{t'=t-R/c}$$

Notation. Put $K(t') = 1 - \beta(t') \cdot \underline{n}(t')$

$[] \equiv$ "evaluated at $t' = t - R/c$ "

$$\phi(\underline{r}_f, t) = \frac{q}{4\pi\epsilon_0 [KR]}$$

$$\underline{A}(\underline{r}_f, t) = \frac{\mu_0 q}{4\pi} \left[\frac{\underline{v}}{KR} \right] \text{ by analogy}$$

The formal simplicity of these "Lienard-Wiechert" potentials for a moving charge is deceptive, because of the tricky implicit dependence that is concealed in the "retardation bracket".

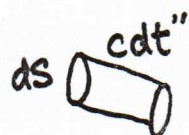
$\underline{v}$, $\underline{n}$ and R will all vary with time

must extract $t' = t - R/c$ and $\left. \begin{matrix} \underline{v}(t') \\ \underline{n}(t') \\ \underline{R}(t') \end{matrix} \right\} \text{ consistently.}$

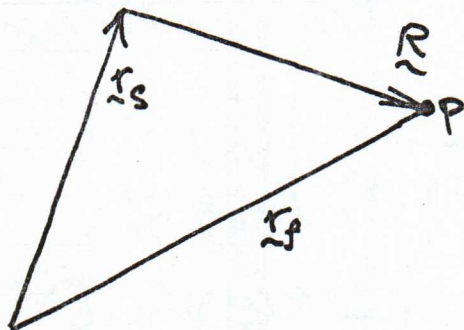
The nastiness of this becomes clear if we differentiate the L-W potentials to derive the $\underline{E}$ and $\underline{B}$ due to general moving charge.

Note that the factor $\frac{1}{K}$ is classical, even though v/c is involved

To visualise what's happened here, consider a finite charge of charge density ρ , sufficiently small tho that R is ~const. ^{over it}



$dS \perp R$
for convenience

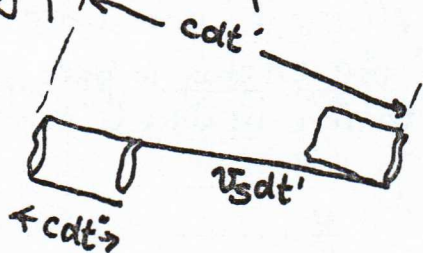


"Information-gathering" sphere collapses onto P at c.

If charge density at r_s was stationary, would take time dt'' to cross it. Invenories charged volume as $dV'' = dS \cdot cdt''$

i.e. compute $\phi = \frac{\rho dS \cdot cdt''}{4\pi\epsilon_0 R}$ (static) ~~$\frac{\rho dS \cdot cdt''}{4\pi\epsilon_0 R}$~~ ($R \gg cdt''$)

Now let charge density move at $v_s(t')$. While the sphere collapses across it, it moves, so contributes to the census of potential for a time $dt' \neq dt''$.



$$cdt' = cdt'' + (v_s(t') \cdot \underline{n}) dt'$$

$$dt' = \frac{dt''}{1 - \beta \cdot \underline{n}} \bigg|_{t' = t - R/c}$$

So what we inventory is $\phi = \frac{\rho dS \cdot cdt'}{4\pi\epsilon_0 R} \bigg|_{t' = t - R/c}$ $R \gg cdt'$

$$= \frac{\phi_{\text{static}}}{K} \bigg|_{t' = t - R/c}$$

This "moving-census" effect is thus "purely classical"

Differentiating the Liénard-Wiechert potentials

To find $\underline{E}$ and $\underline{B}$ at a field point $\underline{r}_f$ due to a moving charge at source point $\underline{r}_s(t)$, we must evaluate

$$\underline{E} = -\underline{\nabla}_f \phi - \frac{\partial \underline{A}}{\partial t} \quad \underline{B} = \underline{\nabla}_f \times \underline{A}$$

$$\text{where: } \phi = \frac{Q}{4\pi\epsilon_0 [KR]} = \frac{Q}{4\pi\epsilon_0} \left(\frac{1}{1 - \beta(t_-) \cdot \underline{n}(t_-)} \right) \left(\frac{1}{|\underline{r}_f - \underline{r}_s(t_-)|} \right)$$

$$\underline{A} = \frac{\mu_0 Q}{4\pi} \left[\frac{\underline{v}}{KR} \right] = \frac{\mu_0 Q}{4\pi} \left(\frac{\underline{v}(t_-)}{1 - \beta(t_-) \cdot \underline{n}(t_-)} \right) \left(\frac{1}{|\underline{r}_f - \underline{r}_s(t_-)|} \right)$$

$$t_- = t - \frac{|\underline{r}_f - \underline{r}_s(t_-)|}{c} = t - \frac{[R]}{c}$$

$\underline{\nabla}_f$ is taken with respect to field point coordinates $\underline{r}_f$ at a fixed observation time t

$\frac{\partial}{\partial t}$ is taken with respect to observation time at a fixed field point $\underline{r}_f$.

The main complexity in this is that small variations in $\underline{r}_f$ at constant t produce variations in the retarded time t_- at which contributions to the potentials are to be evaluated. Thus derivatives with respect to coordinates such as x_f, y_f, z_f at constant t have two components:

$$\left(\frac{\partial}{\partial x_f} \right) = \left(\frac{\partial}{\partial x_f} \right)_{t_- \text{ const}} + \left(\frac{\partial t_-}{\partial x_f} \right) \left(\frac{\partial}{\partial t_-} \right)_{x_f \text{ const}}$$

We can therefore write:

$$\nabla_f = (\nabla_f)_{t_{\text{const}}} + (\nabla_f t_-) \left(\frac{\partial}{\partial t_-} \right)_{r_f \text{ const}}$$

It will make the notation less cluttered in what follows if we drop the subscript f on the ∇ , remembering though that spatial differentiation is always with respect to field point coordinates in these equations.

$$\begin{aligned} \text{Now consider } \nabla_f t_- &= \nabla \left(t - \frac{[R]}{c} \right) \\ &= -\frac{1}{c} \nabla [R] \quad \text{as } t \text{ does not depend on } r_f. \end{aligned}$$

From above, we can expand this to

$$\nabla t_- = -\frac{1}{c} \left\{ (\nabla [R])_{t_{\text{const}}} + (\nabla t_-) \left(\frac{\partial [R]}{\partial t_-} \right)_{r_f \text{ const}} \right\}$$

$$\text{i.e. } c \nabla t_- = -(\nabla [R])_{t_{\text{const}}} - (\nabla t_-) \cdot (-[v \cdot \hat{n}])$$

$$\nabla t_- (c - [v \cdot \hat{n}]) = -(\nabla [R])_{t_{\text{const}}} = -[\dot{R}]$$

$$\text{i.e. } \nabla t_- = -\frac{1}{c} \frac{[\dot{R}]}{(1 - [v \cdot \hat{n}])} = -\frac{1}{c} \left[\frac{\dot{R}}{\kappa} \right]$$

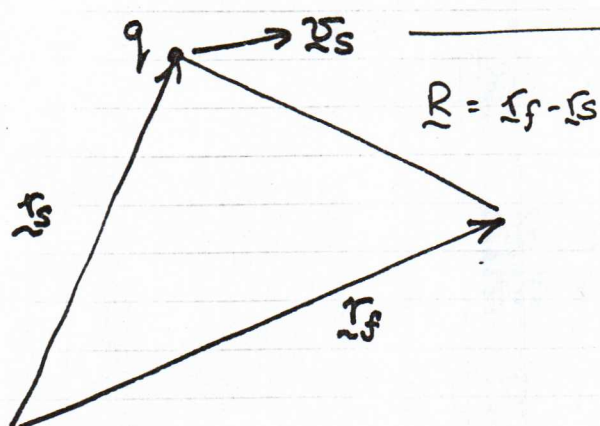
We can now end the implicitness by putting

$$\boxed{\nabla = \nabla_f = (\nabla)_{t_{\text{const}}} - \frac{1}{c} \left[\frac{\dot{R}}{\kappa} \right] \left(\frac{\partial}{\partial t_-} \right)_{r_f \text{ const}}}$$

$$\text{So, consider } \nabla \phi = (\nabla \phi)_{t_{\text{const}}} - \frac{1}{c} \left[\frac{\dot{R}}{\kappa} \right] \left(\frac{\partial \phi}{\partial t_-} \right)_{r_f \text{ const}}$$

$$= \frac{Q}{4\pi\epsilon_0} \left\{ \nabla \left(\frac{1}{[KR]} \right)_{t_{\text{const}}} - \frac{1}{c} \left[\frac{\dot{R}}{\kappa} \right] \left(\frac{\partial (1/[KR])}{\partial t_-} \right)_{r_f \text{ const}} \right\}$$

Wayside Derivatives in the Lienerd-Wiechert Show



call this $\tilde{v}$, as nothing else has a velocity

$$\frac{\partial \tilde{R}}{\partial t_-} = \frac{\partial}{\partial t_-} (\tilde{r}_f - \tilde{r}_s) = 0 - \frac{\partial \tilde{r}_s}{\partial t_-} = -\tilde{v} \quad \frac{\partial \tilde{R}}{\partial t_-} = -\tilde{v} \quad (1)$$

$$\begin{aligned} \left(\frac{\partial \tilde{R}}{\partial t_-} \right)_{r_f \text{ const}} &= \frac{\partial}{\partial t_-} \left(\sqrt{(x_f - x_s)^2 + (y_f - y_s)^2 + (z_f - z_s)^2} \right)_{r_f \text{ const}} \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{\dots}} \cdot \left(2(x_f - x_s) \frac{dx_s}{dt_-} + 2(y_f - y_s) \frac{dy_s}{dt_-} \dots \right) \\ &= \frac{1}{2} \cdot \frac{1}{\tilde{R}} \cdot 2(\tilde{R} \cdot -\tilde{v}) \quad \left(\frac{\partial \tilde{R}}{\partial t_-} \right)_{r_f \text{ const}} = -\tilde{v} \cdot \tilde{R} \quad (2) \end{aligned}$$

$$\begin{aligned} (\nabla \tilde{R})_{t \text{ const}} &= \left(\frac{\partial}{\partial x_f} \tilde{i} + \frac{\partial}{\partial y_f} \tilde{j} + \frac{\partial}{\partial z_f} \tilde{k} \right) \left(\sqrt{(x_f - x_s)^2 + (\quad)^2 + (\quad)^2} \right) \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{\dots}} \cdot \left(2(x_f - x_s) \tilde{i} + 2(y_f - y_s) \tilde{j} + 2(z_f - z_s) \tilde{k} \right) \\ &= \frac{1}{2} \cdot \frac{1}{\tilde{R}} \cdot 2\tilde{R} = \tilde{R} \quad (\nabla \tilde{R})_{t \text{ const}} = \tilde{R} \quad (3) \end{aligned}$$

$$\begin{aligned} \nabla (\beta \cdot \tilde{R})_{t \text{ const}} &= \frac{1}{c} \nabla (\tilde{v} \cdot \tilde{R}) \\ &= \frac{1}{c} \left(\frac{\partial}{\partial x_f} \tilde{i} + \frac{\partial}{\partial y_f} \tilde{j} + \frac{\partial}{\partial z_f} \tilde{k} \right) \left(\tilde{v}_x (x_f - x_s) + \tilde{v}_y (y_f - y_s) + \tilde{v}_z (z_f - z_s) \right) \end{aligned}$$

$$\text{so } \nabla(\beta \cdot \underline{r}) = \frac{1}{c} (v_x \cdot \underline{i} + v_y \cdot \underline{j} + v_z \cdot \underline{k})$$

$$\left(\nabla(\beta \cdot \underline{r}) \right)_{t_{\text{const}}} = \beta \quad (4)$$

$$\begin{aligned} \frac{\partial}{\partial t_-} (\beta \cdot \underline{r}) &= \frac{\partial \beta}{\partial t_-} \cdot \underline{r} + \beta \cdot \frac{\partial \underline{r}}{\partial t_-} \\ &= \dot{\beta} \cdot \underline{r} + \beta \cdot (-\underline{v}) \\ &= \dot{\beta} \cdot \underline{r} - \beta \cdot \underline{v} \end{aligned} \quad (5)$$

Note also that $\frac{\partial R}{\partial t_-} = \nabla_s R \cdot \frac{\partial \underline{r}_s}{\partial t_-} = -\underline{n} \cdot \underline{v}$
[same as #2]

Both differentiations of $1/[KR]$ are best made by chain differentiation:

$$\nabla\left(\frac{1}{[KR]}\right) = -\frac{1}{[K^2 R^2]} \nabla[KR] = -\frac{1}{[K^2 R^2]} \nabla[R - \beta \cdot \underline{R}]$$

$$= -\frac{1}{[K^2 R^2]} [\underline{n} - \beta]$$

$$\frac{\partial [KR]}{\partial t_-} = -\frac{1}{[K^2 R^2]} \frac{\partial [KR]}{\partial t_-} = -\frac{1}{[K^2 R^2]} \left(\frac{\partial [R]}{\partial t_-} - \frac{\partial [\beta \cdot \underline{R}]}{\partial t_-} \right)$$

$$= -\frac{1}{[K^2 R^2]} \left(-[\underline{v} \cdot \underline{n}] - [\beta \cdot \frac{\partial \underline{R}}{\partial t_-}] - \left[\frac{\partial \beta}{\partial t_-} \cdot \underline{R} \right] \right)$$

$$= \frac{1}{[K^2 R^2]} [\underline{v} \cdot \underline{n} + \beta \cdot (-\underline{v}) + \dot{\beta} \cdot \underline{R}]$$

The last term is the eventual key to radiation effects. It contributes a factor varying as $\dot{\beta}/R$ to the fields, i.e. AN R^{-1} field proportional to the acceleration of the charge.

Writing out the differentiations in full, we now have

$$\nabla \phi = \frac{Q}{4\pi\epsilon_0} \left[\frac{\beta - \underline{n}}{[K^2 R^2]} - \frac{\underline{n}}{cK} \frac{1}{K^2 R^2} (\underline{v} \cdot \underline{n} - \beta \cdot \underline{v} + \dot{\beta} \cdot \underline{R}) \right]$$

$$= \frac{Q}{4\pi\epsilon_0 [K^3]} \left[\frac{K(\beta - \underline{n}) - \underline{n}(\beta \cdot \underline{n}) + \beta^2 \underline{n}}{R^2} - \frac{\underline{n}(\dot{\beta} \cdot \underline{n})}{cR} \right]$$

$$= \frac{Q}{4\pi\epsilon_0 [K^3]} \left[\frac{(1 - \beta \cdot \underline{n})(\beta - \underline{n}) - \underline{n}(\beta \cdot \underline{n}) + \beta^2 \underline{n}}{R^2} - \frac{\underline{n}(\dot{\beta} \cdot \underline{n})}{cR} \right]$$

$$\nabla \phi = \frac{Q}{4\pi\epsilon_0 [K^3]} \left[\frac{\beta - \underline{n} - \beta(\beta \cdot \underline{n}) + \beta^2 \underline{n}}{R^2} - \frac{\underline{n}(\dot{\beta} \cdot \underline{n})}{cR} \right]$$

The second contribution to $\underline{E}$ comes from $\underline{A} = \frac{\partial A}{\partial t} = \frac{\partial A}{\partial t_-} \cdot \left(\frac{\partial t_-}{\partial t} \right)_{r, \text{const}}$

$$\text{First consider } \frac{\partial t_-}{\partial t} = \frac{\partial}{\partial t} \left(t - \frac{[R]}{c} \right) = 1 - \frac{1}{c} \frac{\partial [R]}{\partial t} = 1 - \frac{1}{c} \frac{\partial [R]}{\partial t_-} \frac{\partial t_-}{\partial t}$$

$$\text{i.e. } \frac{\partial t_-}{\partial t} = 1 + [\beta \cdot \underline{n}] \frac{\partial t_-}{\partial t},$$

So we have $\frac{\partial t_-}{\partial t} = \frac{1}{1 - [\beta \cdot \underline{n}]}$, i.e. $\boxed{\frac{\partial t_-}{\partial t} = \frac{1}{[K]}}$

$$\begin{aligned} \text{Hence } \underline{\dot{A}} &= \frac{1}{[K]} \frac{\partial \underline{A}}{\partial t_-} = \frac{\mu_0 Q}{4\pi [K]} \frac{\partial}{\partial t_-} \left[\frac{\underline{v}}{KR} \right] \\ &= \frac{\mu_0 Q}{4\pi [K]} \left[\frac{\dot{\underline{v}}}{KR} + \underline{v} \frac{\partial}{\partial t_-} \left(\frac{1}{KR} \right) \right] \\ &= \frac{\mu_0 Q}{4\pi [K]} \left[\frac{\dot{\underline{v}}}{KR} + \frac{\underline{v}}{K^2 R^2} (\underline{v} \cdot \underline{n} - \beta \cdot \underline{v} + \dot{\beta} \cdot \underline{R}) \right] \end{aligned}$$

Putting $\mu_0 = 1/\epsilon_0 c^2$ and taking the c factors into the v 's where possible:

$$\begin{aligned} \underline{\dot{A}} &= \frac{Q}{4\pi \epsilon_0 [K]} \left[\frac{\dot{\underline{\beta}}}{cKR} + \frac{\beta(\beta \cdot \underline{n}) - \beta\beta^2}{K^2 R^2} + \frac{\beta(\dot{\beta} \cdot \underline{n})}{K^2 cR} \right] \\ &= \frac{Q}{4\pi \epsilon_0 [K^3]} \left[\frac{\beta(\beta \cdot \underline{n}) - \beta\beta^2}{R^2} + \frac{K\dot{\underline{\beta}} + \beta(\dot{\beta} \cdot \underline{n})}{cR} \right] \end{aligned}$$

Again note that the terms in R^{-1} and $\dot{\beta}$ are all together

Collecting together $\underline{E} = -\underline{\nabla}\phi - \underline{\dot{A}}$ we get

$$\begin{aligned} \underline{E} &= \frac{Q}{4\pi \epsilon_0 [K^3]} \left[\frac{\underline{n} - \beta + \beta(\beta \cdot \underline{n}) - \underline{n}\beta^2 - \beta(\beta \cdot \underline{n}) + \beta\beta^2}{R^2} + \frac{\underline{n}(\dot{\beta} \cdot \underline{n}) - K\dot{\underline{\beta}} - \beta(\dot{\beta} \cdot \underline{n})}{cR} \right] \\ &= \frac{Q}{4\pi \epsilon_0 [K^3]} \left[\frac{(\underline{n} - \beta)(1 - \beta^2)}{R^2} + \frac{(\underline{n} - \beta)(\dot{\beta} \cdot \underline{n}) - K\dot{\underline{\beta}}}{cR} \right] \end{aligned}$$

This can be written:

$$\underline{E} = \frac{Q}{4\pi \epsilon_0 [K^3]} \left[\frac{\underline{n} - \beta}{\gamma^2 R^2} + \frac{\underline{n} \times \{(\underline{n} - \beta) \times \dot{\underline{\beta}}\}}{cR} \right]$$

where $\gamma = \frac{1}{\sqrt{1 - \beta^2}}$ is the usual Lorentz factor

We then find $\underline{B}$ from $\underline{B} = \nabla \times \underline{A} = \left(\nabla \times \underline{A} \right)_{t_{\text{const}}} + \nabla t_- \times \left(\frac{\partial \underline{A}}{\partial t_-} \right)_{t_{\text{const}}}$

∇t_- and $\frac{\partial \underline{A}}{\partial t_-}$ were evaluated on the way to calculating $\underline{E}$.

$$\text{To find } \left(\nabla \times \underline{A} \right)_{t_{\text{const}}} = \frac{\mu_0 Q}{4\pi} \left[\frac{1}{KR} \nabla \times \underline{r} + \nabla \left(\frac{1}{KR} \right) \times \underline{r} \right]$$

Now $\nabla \times \underline{r} = 0$ and $\nabla \left(\frac{1}{KR} \right)$ was evaluated on the way to calculating $\underline{E}$.

Substituting in the known forms for ∇t_- , $\frac{\partial \underline{A}}{\partial t_-}$ and $\nabla \left(\frac{1}{KR} \right)$, we eventually get:

$$\underline{B} = \frac{Q}{4\pi\epsilon_0 c [K^3]} \left[\frac{\underline{r} \times \underline{a}}{\gamma^2 R^2} + \frac{\underline{r} \times (\underline{a} \times \{(\underline{r} - \underline{\beta}) \times \dot{\underline{\beta}}\})}{cR} \right]$$

i.e. $\underline{B} = \frac{[\underline{a}] \times \underline{E}}{c}$

Note that 1) ALL quantities in square brackets are to be evaluated at the retarded time t_-

- 2) these expressions are EXACT, even for $v \sim c$.
- 3) terms in $1/R^2$ are independent of acceleration ($\dot{\underline{\beta}}$)
- 4) terms in $1/R$ all contain acceleration
- 5) $\underline{E}$ and $\underline{B}$ are always perpendicular to each other
- 6) both are perpendicular to the RETARDED vector $[\underline{r}]$ from the charge to the field point

These results for $\underline{E}$ and $\underline{B}$ from a moving charge form the basis of all calculations of the fields from general charge assemblies. It has been shown by FRISCH and WILETS (Am. J. Phys., 24, 574 (1956)) that these fields can be obtained from minimal postulates of 1) Gauss's Law, 2) charge conservation and 3) the principles of Special Relativity.

We now consider some special cases to explore the nature of these general fields.

7) Fields of Unaccelerated Particles ($\dot{\beta} = 0$) "VELOCITY FIELDS"

For particles travelling in straight lines at constant velocity, $\dot{\beta} = 0$ and the $1/R$ terms vanish:

$$\begin{aligned}\underline{E} &\rightarrow \frac{Q}{4\pi\epsilon_0\gamma^2} \left[\frac{\underline{R} - \underline{\beta}}{K^3 R^2} \right] \\ \underline{B} &\rightarrow \frac{\mu_0 Q}{4\pi\gamma^2} \left[\frac{\underline{v} \times \underline{R}}{K^3 R^2} \right]\end{aligned}$$

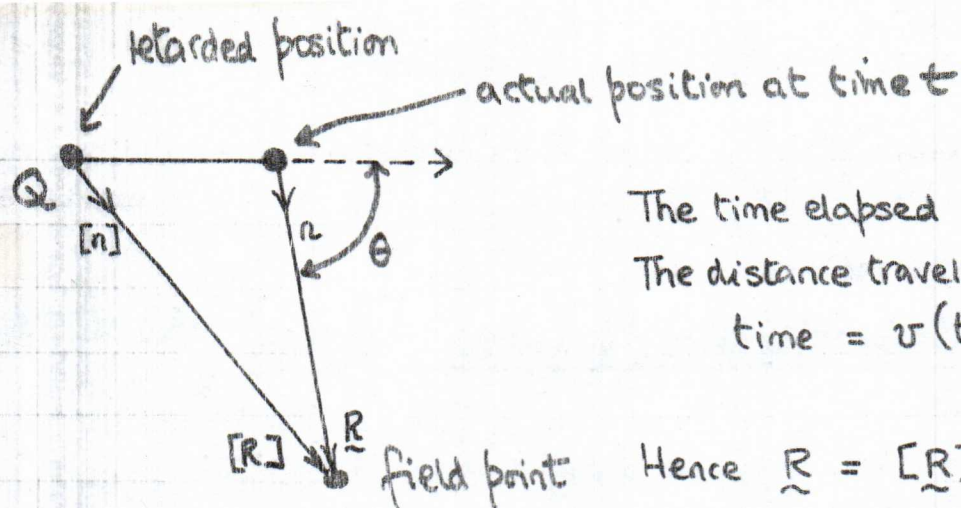
(using $\mu_0 = \frac{1}{\epsilon_0 c^2}$)

Note that in the extreme low-velocity limit, $\beta \rightarrow 0$, $K \rightarrow 1$, $\gamma \rightarrow 1$, these convert to the retarded Coulomb and retarded Biot-Savart fields, i.e. the low-velocity limit consists of the STATIC fields $\underline{E}$ and $\underline{B}$ (associated with Q and $Q\underline{v}$) propagating to the field point with velocity c (to give the "retardation").

For finite velocities, these $\underline{E}$ and $\underline{B}$ expressions are modified versions of the Coulomb and Biot-Savart fields. Consider $\underline{E}$ for illustration:

- 1) It has modified directionality ($\underline{R} - \underline{\beta}$)
- 2) It has modified symmetry (K^{-3}), and is no longer spherically symmetric.

The $(\underline{R} - \underline{\beta})$ term implies that the $\underline{E}$ -field appears to diverge from the ACTUAL position of the charge at time t , NOT from the retarded position. To see this, consider the following diagram:



The time elapsed $t - [t] = \frac{[R]}{c}$

The distance travelled by the charge in this time $= v(t - [t]) = \beta[R]$

Hence $\underline{R} = [R] - \beta[R]$
 $= [R][1 - \beta]$

i.e. the $\underline{E}$ -field of the moving charge (|| to $1 - \beta$) appears to lie along the unretarded vector $\underline{R}$.

- although the "news" of the field propagates from the retarded position of the charge, the lines of electric force point outwards from its actual instantaneous position (due to interference).

A little algebra shows (~~Jackson p. 158~~, Marion, p. 206) that the modified $\underline{E}$ -field can be written in terms of the instantaneous $\underline{R}$ and θ (see diagram) as follows:

$$\underline{E} = \frac{Q \underline{n}}{4\pi\epsilon_0 \gamma^2 R^2 (1 - \beta^2 \sin^2 \theta)^{3/2}}$$

a) In directions AHEAD of the charge, $\underline{\theta} = 0$ and $E_{||} = \frac{Q \underline{n}}{4\pi\epsilon_0 \gamma^2 R^2}$

i.e. the field strength is DECREASED relative to the static case by a factor $1/\gamma^2$.

b) In directions AT RIGHT ANGLES to the charge's motion, $\underline{\theta} = 90^\circ$

$$E_{\perp} = \frac{Q \underline{n}}{4\pi\epsilon_0 \gamma^2 R^2 (1 - \beta^2)^{3/2}} = \frac{Q \underline{n} \gamma^3}{4\pi\epsilon_0 \gamma^2 R^2} = \frac{Q \underline{n} \gamma}{4\pi\epsilon_0 R^2}$$

i.e. the field strength is INCREASED relative to the static case by a factor γ .

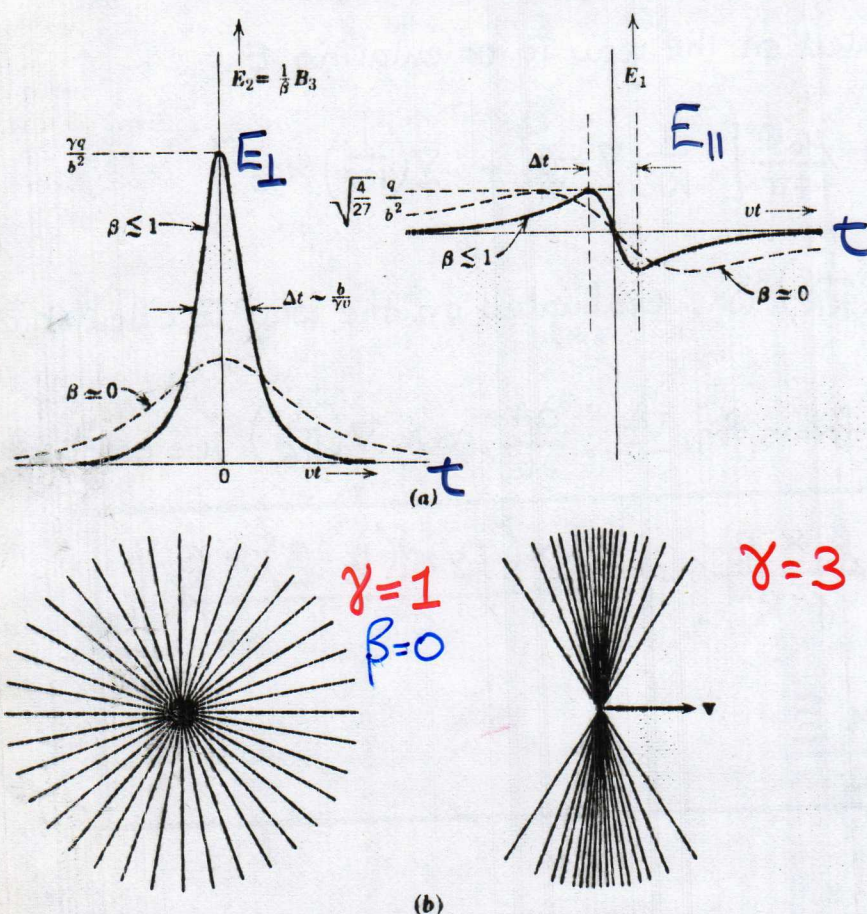


Fig. 11.9 Fields of a uniformly moving charged particle. (a) Fields at the observation point P in Fig. 11.8 as a function of time. (b) Lines of electric force for a particle at rest and in motion ($\gamma=3$).

instantaneous *present position* of the charge in the laboratory. From (11.152) we see that $E_1/E_2 = -vt/b$. Reference to Fig. 11.8 shows that the electric field is thus directed along $\mathbf{n}$, a unit radial vector from the charge's present position to the observation point, just as for a static Coulomb field. By expressing the denominator in (11.152) in terms of r , the radial distance from the present position to the observer, and the angle $\psi = \cos^{-1}(\mathbf{n} \cdot \hat{\mathbf{v}})$ shown in Fig. 11.8, we obtain the electric field in terms of the charge's present position:

$$\mathbf{E} = \frac{q\mathbf{r}}{r^3 \gamma^2 (1 - \beta^2 \sin^2 \psi)^{3/2}} \quad (11.154)$$

The field is radial, but the lines of force are isotropically distributed only for

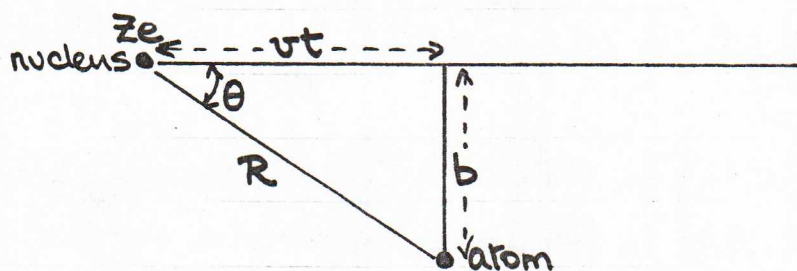
In the relativistic limit $\beta \rightarrow 1$, $\gamma \rightarrow \text{large} \gg 1$, the field becomes entirely transverse to the particle's motion. As $\underline{B} = [\underline{n}] \times \underline{E}/c$ always, we see that THE FIELDS OF A HIGHLY RELATIVISTIC CHARGE IN UNIFORM MOTION ADOPT THE CONFIGURATION OF A PLANE WAVE TRAVELLING WITH THE PARTICLE. (see diagram in Jackson, p. 555)

Note that these are NOT radiation fields, however, as they fall off as $1/R^2$ and cannot propagate energy to arbitrarily large distances.

Sufficiently close to the path of the particle however, a test particle would encounter a PULSE of fields which were mainly transverse to the travelling particle's motion, and which would have many of the characteristics of a pulse of radiation. (see Jackson p. 555).

It is sometimes convenient to think of this pulse as an assembly of VIRTUAL PHOTONS travelling with the moving relativistic particle.

($v \rightarrow c$)
e.g. consider an analysis of the ability of a fast nucleus to produce ionization near its track through a material medium.



b = closest approach of Ze to atom.

Because the field due to the nucleus { falls off as $1/R^2$
{ and is biased to the transverse direction
the main component of the E-field at the atom is

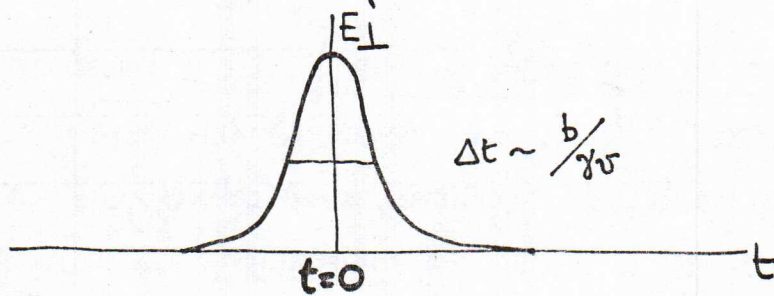
$$E_{\perp} = \frac{Ze \sin \theta}{4\pi\epsilon_0 R^2 (1 - \beta^2 \sin^2 \theta)^{3/2}}$$

Now $\sin \theta = b/R$ and $R = \sqrt{b^2 + v^2 t^2}$

So we can write
$$E_{\perp}(t) = \frac{ze b}{4\pi\epsilon_0 \gamma^2 (b^2 + v^2 t^2)^{3/2} (1 - \beta^2 \frac{b^2}{b^2 + v^2 t^2})^{3/2}}$$
$$= \frac{ze b}{4\pi\epsilon_0 \gamma^2 (v^2 t^2 + b^2 (1 - \beta^2))^{3/2}}$$

$$E_{\perp}(t) = \frac{ze b}{4\pi\epsilon_0 \gamma^2 (v^2 t^2 + b^2/\gamma^2)^{3/2}} \text{ at the atom}$$

This is a pulse of perpendicular field centered on the time at which the nucleus passes the atom



ie at v ,
not at
c

This will be accompanied by a pulse of $B_{\perp}$ of the same shape, with $B_{\perp}$ to E . It is useful to think of this pulse as an assembly of virtual photons moving with the particle whose distribution over frequency is that of the spectrum of the pulse:

$$e_{\perp}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_{\perp}(t) e^{i\omega t} dt$$

$e_{\perp}(\omega)$ contains all the information about the frequency behavior of $E_{\perp}(t)$. To convert this to frequency information about the energy in the pulse consider the Poynting Vector amplitude

$$|\underline{S}| = |\underline{E} \times \underline{H}| = \frac{dW}{dt dA} = \frac{dP}{dA}$$

where dW is the energy crossing ~~area~~ area dA ⁱⁿ ~~time~~ time dt in the fields. ($dP = dW/dt$ is the power ^{crossing} ~~area~~ area dA).

$$\text{Now } \frac{dW}{dA dt} = |\underline{E} \times \underline{H}| = \left| \underline{E} \times \frac{\underline{B}}{\mu_0} \right| = \left| \underline{E} \times \frac{(\underline{E}) \times \underline{E}}{\mu_0 c} \right|$$

$$\text{i.e. } \frac{dW}{dA dt} = \frac{E^2}{\mu_0 c} \quad (\text{also} = \epsilon_0 c E^2)$$

Hence the total energy per unit area in the ^{entire} pulse is

$$\frac{dW}{dA} = \frac{1}{\mu_0 c} \int_{-\infty}^{+\infty} E_{\perp}^2(t) dt$$

$$= \frac{2\pi}{\mu_0 c} \int_{-\infty}^{+\infty} |e_{\perp}(\omega)|^2 d\omega$$

from Parseval's Theorem for F.T.'s
 $\int_{-\infty}^{\infty} E^2(t) dt = 2\pi \int_{-\infty}^{\infty} |e(\omega)|^2 d\omega$

As $E_{\perp}(t)$ will be real function of time, $e_{\perp}(\omega)$ will be symmetric and $e_{\perp}(-\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} E_{\perp}(t) e^{-i\omega t} dt = e_{\perp}(\omega)$

$$\text{Hence } \frac{dW}{dA} = \frac{4\pi}{\mu_0 c} \int_0^{\infty} |e_{\perp}(\omega)|^2 d\omega$$

so we can identify the energy per unit area per unit frequency carried by the pulse as:

$$\underline{\frac{dW}{dA d\omega} = \frac{4\pi}{\mu_0 c} |e_{\perp}(\omega)|^2}$$

Note general validity of this

Note that this is the energy ^{flux} ~~density~~ carried by the entire pulse. It is not "per unit time", and to write "per unit time" would violate the uncertainty principle ~~that~~ $\Delta\omega \Delta t > 1$ for F.T.'s. To relate this to energy density ^{$u(\omega)$} in the pulse per unit frequency we use $dW = du \cdot dA \cdot dt$, $du = u(\omega) d\omega$

$$\text{i.e. } \underline{u(\omega) = \frac{4\pi}{\mu_0 c} |e_{\perp}(\omega)|^2}$$

note that for real photons this would be $\frac{4\pi}{\mu_0 c^2} |e_{\perp}(\omega)|^2$

Hence, for our case

$$e_{\perp}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{ze b}{4\pi\epsilon_0 \gamma^2 (v^2 t^2 + b^2/\gamma^2)^{3/2}} \cdot e^{i\omega t} dt$$

$$e_{\perp}(\omega) = \frac{ze b}{8\pi^2 \epsilon_0 \gamma^2} \int_{-\infty}^{+\infty} \frac{e^{i\omega t}}{(v^2 t^2 + b^2/\gamma^2)^{3/2}} dt$$

Substitute $y = \frac{\gamma v t}{b}$, then this becomes

$$e_{\perp}(\omega) = \frac{ze}{8\pi^2 \epsilon_0 b v} \int_{-\infty}^{+\infty} \frac{e^{i(\omega b/\gamma v)y}}{(y^2 + 1)^{3/2}} dy$$

This is a known integral giving the modified Bessel function of order unity.

$$e_{\perp}(\omega) = \frac{ze}{4\pi^2 \epsilon_0 b v} \left\{ \left(\frac{\omega b}{\gamma v} \right) K_1 \left(\frac{\omega b}{\gamma v} \right) \right\}$$

We can deal with the asymptotic forms: $K_1(x) \rightarrow \frac{1}{2x} \quad x \ll 1$
 $\rightarrow \sqrt{\frac{\pi}{2x}} e^{-x} \quad x \gg 1$

So, if $x = \frac{\omega b}{\gamma v} \gg 1$, $e_{\perp}(\omega)$ falls off exponentially with $\frac{\omega b}{\gamma v}$

if $x = \frac{\omega b}{\gamma v} \ll 1$, $e_{\perp}(\omega) \rightarrow \frac{ze}{4\pi^2 \epsilon_0 b v}$

$$u(\omega) \rightarrow \frac{4\pi}{\mu_0 c} \cdot \frac{z^2 e^2}{16\pi^4 \epsilon_0^2 b^2 v^2}$$

$$u(\omega) \rightarrow \frac{z^2 e^2 c}{4\pi^3 \epsilon_0 b^2 v^3} \quad \text{independent of } \omega$$

Note that if replacing the entire pseudo-plane wave field associated with a uniformly-moving charge by an equivalent energy-frequency distribution we would say

$$U(\omega) = \int_0^\infty u(\omega) \cdot v \cdot 2\pi b \, db$$

$$\sim \int_0^{\gamma v/\omega} \frac{z^2 e^2 c}{4\pi^2 \epsilon_0 b^2 v^3} \cdot v \cdot 2\pi b \cdot db = \int_0^{\gamma v/\omega} \frac{z^2 e^2 c}{2\pi^2 \epsilon_0 v^2} \cdot \frac{db}{b}$$

The integral diverges for $b_{\min} \rightarrow 0$, and we would have to take a minimum b at some value that ascribed an arbitrary lower cutoff to the classical Coulomb field (per from Q. Mech.)

Then $U(\omega) = \frac{2}{\pi} \cdot \frac{z^2 e^2 c}{4\pi \epsilon_0 v^2} \ln\left(\frac{\gamma v}{\omega b_{\min}}\right)$ $b_{\min} \sim$ "radius of atom" for ionization

and the number of "equivalent photons" available for ionization is $N(\omega)$ where:

$$U(\omega) d\omega = \hbar \omega N(\omega) d\omega$$

$$N(\omega) = \frac{2}{\pi} \cdot \frac{z^2 e^2}{4\pi \epsilon_0 \hbar c} \cdot \frac{c^2}{v^2} \ln\left(\frac{\gamma v}{\omega b_{\min}}\right) \cdot \frac{1}{\omega}$$

$$N(\omega) = \frac{2\alpha}{\pi} \cdot \frac{c^2}{v^2} \cdot \frac{z^2}{\omega} \ln\left(\frac{\gamma v}{\omega b_{\min}}\right)$$

where α is the fine structure constant $e^2/4\pi\epsilon_0\hbar c \sim 1/137$

As $v \rightarrow c$, $N(\omega) \rightarrow \frac{2\alpha}{\pi} \cdot \frac{z^2}{\omega} \ln\left(\frac{\gamma c}{\omega b_{\min}}\right)$

This equation is useful in relating the probability of processes induced by charged particles acting only through their electromagnetic fields, to the probability of processes induced by radiation - "virtual photon" concept.

Note that the total energy carried by the fields at frequency ω increases with the energy $\gamma m_0 c^2$ of the fast nucleus, but the energy density is most favorable for ionization if a heavy nucleus (high Z) passes slowly close by the atom to be ionized (v and b small)

For more detailed discussion of applications of virtual photon concept to representing fields due to fast charged particles, see Jackson, Sect. 13.2 / 15.4.

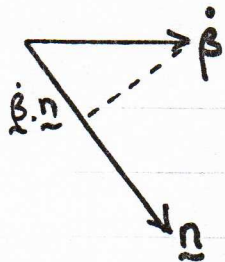
8) Radiation at Low Velocities ($\beta \rightarrow 0, \dot{\beta} \neq 0$)

A very large number of classical radiation problems can be treated in the limit $\beta \rightarrow 0, \gamma \rightarrow 1, k \rightarrow 1, \dot{\beta}$ finite. In this limit

$$\begin{aligned}\underline{E} &\rightarrow [\underline{E}_{\text{Coulomb}}] + \frac{Q}{4\pi\epsilon_0 R c} [\underline{n} \times (\underline{n} \times \dot{\underline{\beta}})] \\ \underline{B} &\rightarrow [\underline{B}_{\text{Biot-Savart}}] + \frac{Q}{4\pi\epsilon_0 R c^2} [\underline{n} \times (\underline{n} \times (\underline{n} \times \dot{\underline{\beta}}))]\end{aligned}$$

The $1/R$ fields contribute to a Poynting Vector $\underline{E} \times \underline{H}$ which falls off as $1/R^2$ and can therefore transfer energy from the accelerated charge to infinity. Unlike the "quasi-plane-wave" fields associated with the fast unaccelerated charges of the previous section, these fields are not bound to their sources and can RADIATE to infinity, causing an energy loss whether or not other charges are in the vicinity of the accelerated source.

From the vector relation $\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\underline{b} - (\underline{a} \cdot \underline{b})\underline{c}$, we have

$$\underline{n} \times (\underline{n} \times \dot{\underline{\beta}}) = (\underline{n} \cdot \dot{\underline{\beta}})\underline{n} - \dot{\underline{\beta}}$$


Evidently $(\underline{n} \cdot \dot{\underline{\beta}})\underline{n} = \dot{\underline{\beta}}_{\parallel}$ ($\parallel$ to the line of sight)

So $(\underline{n} \cdot \dot{\underline{\beta}})\underline{n} - \dot{\underline{\beta}} = -\dot{\underline{\beta}}_{\perp}$ ($\perp$ to the line of sight)

So $\underline{E}$ in the radiation field is proportional to $[\dot{\underline{\beta}}_{\perp}]$
antiparallel to

i.e. the E-fields radiated by an accelerated charge are governed by the component of the acceleration perpendicular to the observer's line of sight. (linear polarization $\parallel$ to $\dot{\underline{\beta}}_{\perp}$)

i.e. THERE IS NO RADIATION IN THE DIRECTION OF $[\dot{\beta}]$

Note also that $\underline{n} \times (\underline{n} \times (\underline{n} \times \dot{\beta})) = -(\underline{n} \times \dot{\beta}) = (\dot{\beta} \times \underline{n})$

so that $\underline{B}$ in the radiation fields is perpendicular to $[\dot{\beta}]$ and to $[\underline{n}]$, as well as to $\underline{E}$.

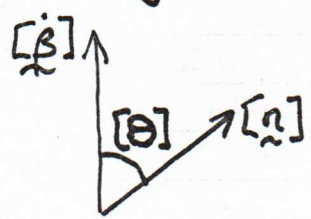
Note also the direction of the Poynting Vector

$$\underline{S} = \underline{E} \times \underline{H} = \underline{E} \times \frac{\underline{B}}{\mu_0} = \underline{E} \times \frac{[\underline{n}] \times \underline{E}}{\mu_0 c} = \frac{E^2 [\underline{n}] - (\underline{E} \cdot [\underline{n}]) \underline{E}}{\mu_0 c}$$

i.e. $\underline{S} = \frac{E^2}{\mu_0 c} [\underline{n}] = \epsilon_0 c E^2 [\underline{n}]$

Notation: $\mu_0 c = \frac{1}{\epsilon_0 c} = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \text{ ohms} = Z_0$ "characteristic impedance of free space"

In angular terms, let $[\theta]$ be angle between $[\dot{\beta}]$ and $[\underline{n}]$

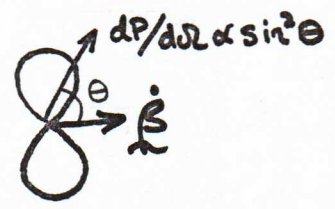


$$\begin{aligned} \beta_{\perp} &= \dot{\beta} \sin \theta \\ |E| &= \frac{Q [\dot{\beta} \sin \theta]}{4\pi \epsilon_0 [R] c} \end{aligned}$$

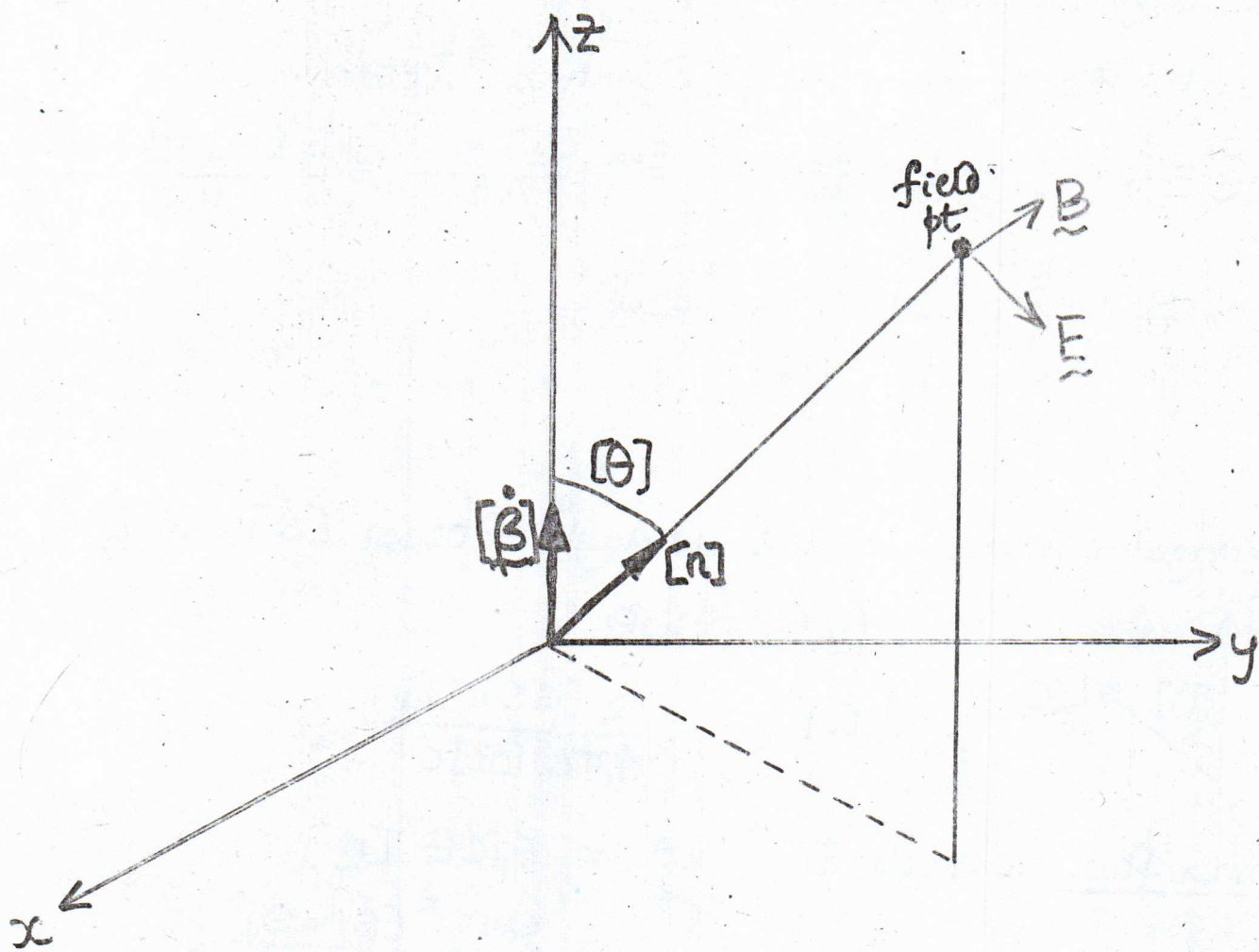
Power flux at angle θ , $\underline{S} = \epsilon_0 c E^2 [\underline{n}]$
 $= \frac{\epsilon_0 c Q^2 [\dot{\beta} \sin \theta]^2}{(4\pi \epsilon_0 c [R])^2} [\underline{n}]$

Power Density (instantaneous power radiated per unit solid angle) is:

$$\frac{dP(\theta)}{d\Omega} = (\underline{S} \cdot \underline{n}) [R^2] = \frac{Q^2 [\dot{\beta}]^2 \sin^2 [\theta]}{16\pi^2 \epsilon_0 c}$$



Because low-velocity radiation is controlled by $\beta_{\perp}$, most elementary radiator has $\sin^2 \theta$ "polar diagram" - underlying directionality of all low-velocity radiating systems.



Configuration of $[\dot{\beta}]$, $[\underline{n}]$, $\underline{E}$ and $\underline{B}$
 in radiation fields of accelerated charge
 at LOW VELOCITIES ($\beta \rightarrow 0$)

Total Power Radiated (over all directions) for low-velocity charge is

$$P = -\frac{d\mathcal{E}}{dt} = \int_{4\pi \text{ ster}} \frac{dP(\theta)}{d\Omega} d\Omega = \frac{Q^2 [\dot{\beta}]^2}{16\pi^2 \epsilon_0 c} \int_0^{2\pi} d\phi \int_0^\pi \sin^2[\theta] \sin[\theta] d\theta$$

$$= \frac{Q^2 [\dot{\beta}]^2}{16\pi^2 \epsilon_0 c} \cdot 2\pi \cdot \frac{4}{3}$$

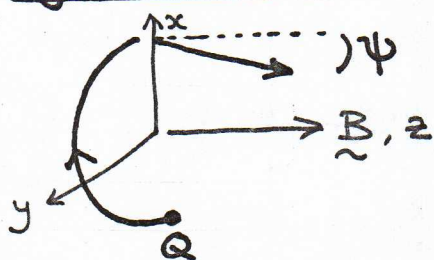
$$P = \frac{Q^2 [\dot{\beta}]^2}{6\pi \epsilon_0 c} = \frac{Q^2 [\dot{v}]^2}{6\pi \epsilon_0 c^3}$$

This useful expression for the total radiated power was given by J.J. Larmor in 1897 and is often called LARMOR'S FORMULA

9) Some Applications

We now consider some specific examples of radiation from low-velocity charges.

a) Cyclotron radiation (at low velocities)



Charge circles $\underline{B}$ at $\omega = \frac{QB}{m}$

$$\left(\frac{mv^2}{r} = QvB \right)$$

CYCLOTRON FREQUENCY $\omega = \frac{v}{r} = \frac{QB}{m}$

Take z axis along $\underline{B}$
field point at point in (x, z) plane at angle ψ to $\underline{B}$
(this defines x and y axes)

$$\text{Then } \underline{\dot{\beta}}(t) = -\frac{QvB}{mc} (\underline{i} \cos \omega t + \underline{j} \sin \omega t)$$

if t is measured from zero as charge crosses $\oplus$ ve x -axis.

Q. What is time-averaged power radiated per unit solid angle as a function of ψ ?

i.e. What is $\left\langle \frac{dP(\psi)}{d\Omega} \right\rangle$?

$\langle \rangle$ - average over period at freq. ω .

Method: $\vec{E}_{\text{rad}} = \frac{Q}{4\pi\epsilon_0 R c} [\vec{n} \times (\vec{n} \times \dot{\vec{\beta}})]$

↑
need this as $f(t, \psi)$

As motion is periodic, time-average over the period will remove the retardation effects, hence need not distinguish $[\]$ quantities.

At $R \gg r$, (far from orbit)

$$\vec{n} \rightarrow -\hat{i} \sin \psi + \hat{k} \cos \psi$$

$$\text{So } (\vec{n} \times \dot{\vec{\beta}}) \rightarrow \left(-\frac{QvB}{mc} \right) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin \psi & 0 & \cos \psi \\ \cos \omega t & \sin \omega t & 0 \end{vmatrix}$$

$$= -\frac{QvB}{mc} \{ -\hat{i} \cos \psi \sin \omega t + \hat{j} \cos \psi \cos \omega t - \hat{k} \sin \psi \sin \omega t \}$$

$$\vec{n} \times (\vec{n} \times \dot{\vec{\beta}}) = \left(-\frac{QvB}{mc} \right) \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin \psi & 0 & \cos \psi \\ -\cos \psi \sin \omega t & \cos \psi \cos \omega t & -\sin \psi \sin \omega t \end{vmatrix}$$

$$= -\frac{QvB}{mc} \{ \hat{i} \omega^2 \psi \cos \omega t - \hat{j} (\omega^2 \psi + \sin^2 \psi) \sin \omega t - \hat{k} (\sin \psi \cos \psi) \cos \omega t \}$$

Hence $\underline{E}_{\text{rad}} = \frac{Q^2 v B}{4\pi\epsilon_0 R m c^2} \left[\underline{i} \cos^2 \psi \cos \omega t + \underline{j} \sin \omega t + \underline{k} (\sin \psi \cos \psi) \cos \omega t \right]$

$$\begin{aligned} \frac{dP(\psi)}{d\Omega} &= R^2 |S| = R^2 \epsilon_0 c |E_{\text{rad}}|^2 \\ &= \frac{Q^4 v^2 B^2}{16\pi^2 \epsilon_0^2 m^2 c^4} \cdot \epsilon_0 c \cdot \{ (\cos^4 \psi + \sin^2 \psi \cos^2 \psi) \cos^2 \omega t + \sin^2 \omega t \} \end{aligned}$$

i.e. $\left\langle \frac{dP(\psi)}{d\Omega} \right\rangle = \frac{Q^4 v^2 B^2}{16\pi^2 \epsilon_0 m^2 c^3} \{ \cos^2 \psi \langle \cos^2 \omega t \rangle + \langle \sin^2 \omega t \rangle \}$

i.e. $\boxed{\left\langle \frac{dP(\psi)}{d\Omega} \right\rangle = \frac{Q^4 v^2 B^2}{32\pi^2 \epsilon_0 m^2 c^3} \{ \cos^2 \psi + 1 \}}$

$$= \frac{Q^2 v^2 \omega^2}{16\pi^2 \epsilon_0 c^3 m} \left\{ 1 - \frac{1}{2} \sin^2 \psi \right\}$$

Note that for $\psi = 0$ or 180° ($\perp$ to the orbit), the radiation is a MAXIMUM:

$$\underline{E}_{\text{rad}} = \frac{Q^2 v B}{4\pi\epsilon_0 R m c^2} \left[\underline{i} \cos \omega t + \underline{j} \sin \omega t \right] \quad \text{CIRCULAR POLARIZATION}$$

For $\psi = \pm 90^\circ$ (in plane of orbit), radiation is a MINIMUM:

$$\underline{E}_{\text{rad}} = \frac{Q^2 v B}{4\pi\epsilon_0 R m c^2} \left[\underline{j} \sin \omega t \right] \quad \text{LINEAR POLARIZATION}$$

If all we wanted was TOTAL POWER radiated, much easier to use Larmor's formula:

$$\begin{aligned}
 P &= -\frac{\partial \Sigma}{\partial t} = \frac{Q^2 [\dot{\beta}]^2}{6\pi\epsilon_0 c} = \frac{Q^2}{6\pi\epsilon_0 c} \left(\frac{QvB}{m} \right)^2 \\
 &= \frac{Q^4 v^2 B^2}{6\pi\epsilon_0 c^3 m^2} \\
 &= \frac{Q^2 \omega^2 v^2}{6\pi\epsilon_0 c^3}
 \end{aligned}$$

This energy loss rate by radiation in a cyclotron must be made up in power supplied by the accelerating fields. Linear accelerators better because no "useless" accelerations are present.

b) Free electron scattering



Consider a free electron, charge q , mass m

exposed to incident $\underline{E}$ from an electromagn. wave

$$|B| = \frac{1}{c} |E| \quad \text{so} \quad |\underline{v} \times \underline{B}| \sim \frac{v}{c} |E|$$

and we can ignore the force on electron due to $\underline{B}$ of wave, unless $v \rightarrow c$ in a period of the field.

Electron experiences acceleration $\underline{\ddot{v}} = \frac{q \underline{E}}{m}$

Radiates a total power $P = \frac{q^4 E^2}{6\pi\epsilon_0 m^2 c^3}$ Summed over all directions

Now if average power density incident is $\langle I \rangle = \epsilon_0 c \langle E^2 \rangle$

So we can write $\langle P \rangle = \frac{q^4}{6\pi\epsilon_0^2 m^2 c^4} \langle I \rangle$ for the "scattered" power.

In the language of cross-sections, the "scattering cross-section" of the single free electron is the area of the incident beam containing a power equal to the power that is scattered, i.e.

$$\sigma = \frac{\langle P \rangle}{\langle I \rangle} = \frac{q^4}{6\pi\epsilon_0^2 m^2 c^4}$$

This is known as the THOMSON CROSS-SECTION for a free electron, and is $\sim 6.65 \times 10^{-29} \text{ m}^2$. It is normally written σ_T .

In terms of the "classical electron radius" r_0 defined by $mc^2 = \frac{q^2}{4\pi\epsilon_0 r_0}$

$$\sigma_T = \frac{8\pi}{3} r_0^2$$

which is the form in which it most commonly written. ($r_0 = 2.8 \times 10^{-15} \text{ m}$)

The free electron scattering cross-section is independent of frequency until we approach high frequencies at which $\hbar\omega \sim mc^2$. At $\hbar\omega \gtrsim mc^2$ we must use the quantum-mechanical Dirac equation for the electron, and the Thomson cross-section is replaced by the Klein-Nishina cross section, whose asymptotic form ($\hbar\omega \gg mc^2$) is

$$\sigma_{KN} = \pi r_0^2 \cdot \frac{mc^2}{\hbar\omega} \left[\frac{1}{2} + \ln\left(\frac{2\hbar\omega}{mc^2}\right) \right]$$

The Thomson cross-section is in good agreement with experiments on scattering of soft X-rays from electrons, or γ -rays from light atoms.

Panofsky and Phillips, Ch 21-6

c) Radiative Damping (see Marion, Sect. 9.9; Jackson, Sect. 17.2)

The radiative loss of energy experienced by an accelerated charge is often modelled by an equivalent radiative reaction force $\underline{F}_r$ which acts such as to dampen the motion of the charge. Ideally we would like to find a force $\underline{F}_r$ such that

$$\underline{F}_r \cdot \underline{v} = \frac{d\mathcal{E}}{dt} \quad \text{at all times.}$$

But for energy loss by radiation we have Larmor's Formula, so we are asking to equate

$$\underline{F}_r \cdot \underline{v} = - \frac{Q^2 \underline{\ddot{v}} \cdot \underline{\ddot{v}}}{6\pi\epsilon_0 c^3}$$

In general motions, $\underline{v}$ and $\underline{\ddot{v}}$ are uncorrelated at any instant, so no form could exist for $\underline{F}_r$ such that this relation would always be true. We can however form an expression for $\underline{F}_r$ which gives an adequate representation of the average energy balance between two times t_1 and t_2

i.e.
$$\int_{t_1}^{t_2} (\underline{F}_r \cdot \underline{v}) dt = - \frac{Q^2}{6\pi\epsilon_0 c^3} \int_{t_1}^{t_2} (\underline{\ddot{v}} \cdot \underline{\ddot{v}}) dt$$

$$= - \frac{Q^2}{6\pi\epsilon_0 c^3} \left\{ (\underline{v} \cdot \underline{\ddot{v}}) \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} (\underline{v} \cdot \underline{\dddot{v}}) dt \right\} \quad \begin{array}{l} \text{integrating} \\ \text{by parts} \end{array}$$

The point is that because $\underline{v}$ and $\underline{\ddot{v}}$ are uncorrelated, the average value of $(\underline{v} \cdot \underline{\ddot{v}}) \Big|_{t_1}^{t_2}$ must be zero. We can equally well choose $\underline{\ddot{v}}(t)$ to be one Fourier component of a complicated $\underline{\ddot{v}}(t)$, in which case $(\underline{v} \cdot \underline{\ddot{v}}) \Big|_{t_1}^{t_2}$ will be zero if $t_2 - t_1$ is a multiple of the period of the Fourier component.

$$\therefore \int_{t_1}^{t_2} (\underline{F}_r \cdot \underline{v}) dt \simeq + \frac{Q^2}{6\pi\epsilon_0 c^3} \int_{t_1}^{t_2} (\underline{v} \cdot \underline{\ddot{v}}) dt$$

The average effects of radiative losses can therefore be modelled by a radiative reaction force

$$\underline{F}_r = + \frac{Q^2 \ddot{\underline{v}}}{6\pi\epsilon_0 c^3}$$

For harmonic oscillations at frequency ω , the radiative reaction is therefore

$$\underline{F}_r = \frac{Q^2 \ddot{\underline{v}}}{6\pi\epsilon_0 c^3} = \frac{Q^2 \ddot{\underline{x}}}{6\pi\epsilon_0 c^3} = - \frac{Q^2 \omega^2 \underline{x}}{6\pi\epsilon_0 c^3}$$

$$\underline{F}_r = -m\Gamma_r \underline{x} \quad \text{where} \quad \Gamma_r = \frac{Q^2 \omega^2}{6\pi\epsilon_0 c^3 m}$$

It will often be convenient to include the effects of radiative energy losses in a physical system by including an effective velocity damping Γ_r in the equation of motion of a typical charge in the system.

Note that in the case $\dot{\underline{v}} = \text{constant}$ (accelerated motion of a charge in a straight line) there is clearly energy radiated, although the general expression for the radiative reaction gives $\underline{F}_r = 0$ ($\ddot{\underline{v}} = 0$). Where then does the energy of the radiation come from?

d) Scattering by bound electrons

Now consider an electromagnetic wave ^{of freq ω} incident on an electron that is bound in a harmonic potential with characteristic frequency ω_0 .

Suppose the radiation travels in the z -direction, and is linearly polarized with $\underline{E} \parallel$ to the x -axis. Then we can write the equation of motion of the electron in the x -direction:

Trouble with Radiative Damping Term?

Consider the free electron equation $m\ddot{\underline{r}} = \frac{Q^2 \ddot{\underline{r}}}{6\pi\epsilon_0 c^3} = \frac{Q^2}{6\pi\epsilon_0 c^3} \frac{d\dot{\underline{r}}}{dt}$

$$\left(\text{This gives } \ddot{\underline{r}} - \frac{6\pi\epsilon_0 c^3 \ddot{\underline{r}}}{Q^2} = 0 \right)$$

Solution is $\dot{\underline{r}}(t) = 0$ or $\dot{\underline{r}}(t) = a e^{t/\tau}$ where a is $\dot{\underline{r}}(0)$

↓
reasonable

↓
"runaway"

$$\tau \sim \frac{Q^2}{6\pi\epsilon_0 c^3 m} \sim 10^{-23} \text{ sec}$$

Runaway solⁿ. Says that unless $\dot{\underline{r}}(0) = 0$, electron will instantly get large velocity and shoot off to infinity.

Something drastically wrong!

Note that $(\dot{\underline{r}}, \underline{r})$ would not $\rightarrow 0$ at t_1 and t_2 in Free derivation

Must also limit use of radiative damping term to cases where radiative damping $\rightarrow$ small corrections to motions of charges.

$$\ddot{x} + \Gamma \dot{x} + \omega_0^2 x = \frac{q}{m} E_\omega e^{-i\omega t}$$

Note that the forcing amplitude E_ω need only be the ω -frequency component $e(\omega)$ of an actual general wave $E(t)$

$$E(t) = \int_{-\infty}^{\infty} e(\omega) e^{-i\omega t} d\omega$$

Note also that in general $\Gamma = \Gamma_c + \Gamma_r$ where Γ_c is a COLLISIONAL damping term intrinsic to the binding of the electron. Γ_c and ω_0 would in practice be obtained from quantum theory of atom/molecule providing the binding.

Now note that the diff. eqn. for $x(t)$ is that for forced, damped harmonic motion, solution has known form:

$$x(t) = \frac{q E_\omega e^{-i\omega t}}{m(\omega_0^2 - \omega^2 - i\omega\Gamma)} + \text{transient } (\propto e^{-\Gamma t/2})$$

i.e. the acceleration of the electron in the steady state is:

$$\dot{v}(t) = \ddot{x}(t) = \frac{-\omega^2 q E_\omega e^{-i\omega t}}{m(\omega_0^2 - \omega^2 - i\omega\Gamma)}$$

Recalling our "take the real part" conversion for what the exponential means:

$$\dot{v}(t) = -\frac{\omega^2 q E_\omega}{m} \left\{ \frac{(\omega_0^2 - \omega^2) \cos \omega t + \omega \Gamma \sin \omega t}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2} \right\}$$

To compute the radiation, put this into the Larmor formula and time-average

$$\begin{aligned} \langle P \rangle &= \frac{q^2}{6\pi\epsilon_0 c^3} \cdot \frac{\omega^4 q^2 E_\omega^2}{m^2} \cdot \frac{(\omega_0^2 - \omega^2)^2 \langle \cos^2 \omega t \rangle + 2\omega\Gamma(\omega_0^2 - \omega^2) \langle \sin \omega t \cos \omega t \rangle + \omega^2 \Gamma^2 \langle \sin^2 \omega t \rangle}{((\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2)^2} \\ &= \frac{q^2}{6\pi\epsilon_0 c^3} \cdot \frac{\omega^4 q^2 E_\omega^2}{m^2} \cdot \frac{1}{2} \cdot \frac{1}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2} \end{aligned}$$

Noting that $\langle I \rangle = \epsilon_0 c \langle E^2 \rangle = \frac{1}{2} \epsilon_0 c E_0^2$

$$\boxed{\frac{\langle P \rangle}{\langle I \rangle} = \sigma = \sigma_T \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}}$$

This is the bound-electron equivalent of the Thomson Cross-section for a free electron.

The free-electron case reduces to $\omega_0 = 0$ $\sigma \rightarrow \sigma_T$
 $\Gamma \Rightarrow 0$ ($\Gamma \ll \omega$)

Rayleigh scattering

This is the case where the wave frequencies are well below the characteristic frequencies

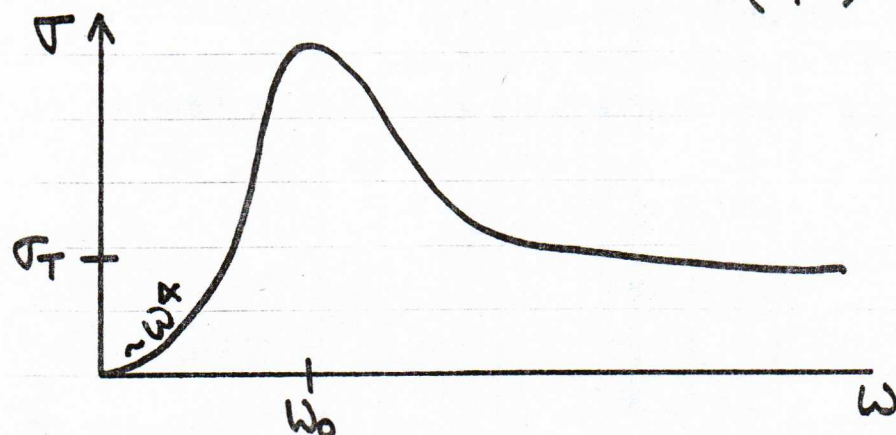
$$\omega \ll \omega_0$$

$$\sigma \rightarrow \sigma_T \cdot \left(\frac{\omega}{\omega_0} \right)^4$$

i.e. Scattered power $\propto \lambda^{-4}$

At high frequencies $\omega \gg \omega_0$, $\sigma \rightarrow \sigma_T$

At resonance, $\omega = \omega_0$ $\sigma = \sigma_T \left(\frac{\omega_0}{\Gamma} \right)^2$ (resonant fluorescence)



In gases, electromagnetic radiation encounters randomly-distributed molecules containing bound electrons. (Electrons are the main scatterers because of the $1/m^2$ term in the scattering cross-section). The scattering is incoherent because of the random distribution of scattering centres, so the scattered power is simply proportional to the number of scatterers per volume, each contributing a power given by the cross-section calculation.

For atmospheric molecules, the appropriate resonances are in the ultraviolet, so visible light obeys Rayleigh scattering, which is much stronger in the blue than in the red, giving a blue sky background.

Larger particles, such as water droplets, produce scattering by diffuse reflection and diffraction of light which removes most of the wavelength dependence — giving generally white clouds.

e) Natural line width of classical unforced oscillator

The radiative reaction imposes a minimum line width on the spectrum of a classical unforced oscillator. To see this, consider a bound harmonic oscillator of characteristic frequency ω_0 , subject to radiative damping Γ and excited from rest at its equilibrium position at $t=0$.

Equation of motion is $\ddot{x} + \Gamma \dot{x} + \omega_0^2 x = 0$

Look for solution in form $\left. \begin{aligned} x &= A e^{\alpha t} \\ \dot{x} &= A \alpha e^{\alpha t} \\ \ddot{x} &= A \alpha^2 e^{\alpha t} \end{aligned} \right\} \begin{aligned} \alpha^2 + \Gamma \alpha + \omega_0^2 &= 0 \\ \downarrow \\ \alpha &= \frac{-\Gamma \pm \sqrt{\Gamma^2 - 4\omega_0^2}}{2} \end{aligned}$

So solution $_{(t>0)}$ is $x(t) \Rightarrow A e^{-\Gamma t/2} e^{\pm i\omega_0 t}$ if $\Gamma^2 \ll 4\omega_0^2$

For electrons, $\Gamma = \frac{Q^2 \omega_0^2}{6\pi \epsilon_0 c^3 m} = 6.25 \times 10^{-24} \omega_0^2$

So $\Gamma \ll 2\omega_0$ if $6.25 \times 10^{-24} \omega_0^2 \ll 2\omega_0$, i.e. $\omega_0 \ll 3 \times 10^{23}$ rad/s.
(below hard γ -rays)

Then consider the component of the electron motion at frequency ω

$$x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} x(t) e^{i\omega t} dt$$

$$= \frac{A}{2\pi} \int_0^{\infty} e^{-i\omega_0 t} e^{-\Gamma t/2} e^{i\omega t} dt$$

$$= \frac{A}{2\pi} \left\{ \frac{1}{i(\omega - \omega_0) - \Gamma/2} \right\} \quad \text{is the amplitude of this cpt.}$$

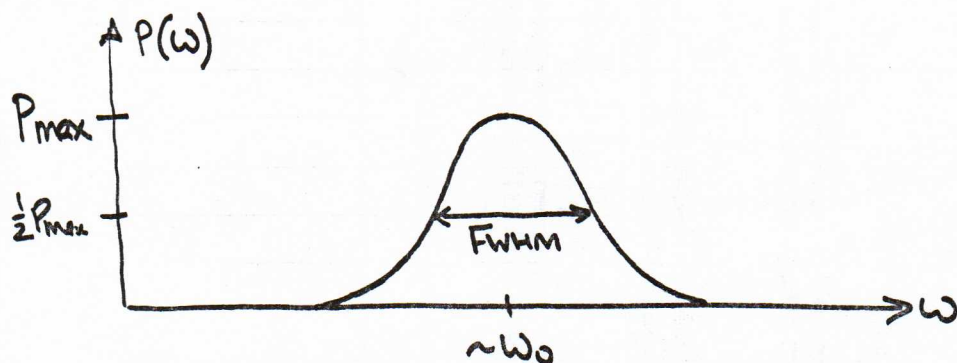
$$\ddot{x}(\omega) = \frac{A}{2\pi} \left\{ \frac{\omega^2}{i(\omega - \omega_0) - \Gamma/2} \right\} \quad \text{is amplitude of acceleration at frequency } \omega.$$

$$= \frac{A\omega^2}{2\pi} \left\{ \frac{i(\omega - \omega_0) - \Gamma/2}{(\omega - \omega_0)^2 + \Gamma^2/4} \right\}$$

i.e. $|\ddot{x}|$ at frequency ω is $\frac{A\omega^2}{2\pi} \cdot \frac{1}{\sqrt{(\omega-\omega_0)^2 + \Gamma^2/4}}$

Put into Larmor's formula: $\langle P \rangle = \frac{Q^2 \langle \ddot{x} \rangle^2}{6\pi\epsilon_0 c^3}$

$$\langle P \rangle = \frac{Q^2 A^2 \omega^4}{6\pi\epsilon_0 c^3 \cdot 4\pi^2} \cdot \frac{1}{2} \frac{1}{(\omega-\omega_0)^2 + \Gamma^2/4}$$



Radiated power peaks at $\omega_{\max} \approx \omega_0$ (actually below ω_0)

$$P = \frac{1}{2} P_{\max} \text{ at } \omega = \omega_0 \pm \Gamma/2$$

$$\text{FWHM} = \Gamma$$

"Natural line width" in frequency is $\Delta\omega = \Gamma \sim \Gamma_r(\omega_0)$

$$= \frac{Q^2 \omega_0^2}{6\pi\epsilon_0 c^3 m}$$

in wavelength is $\Delta\lambda = \frac{2\pi c}{\omega_0^2} \Delta\omega$

$$= \frac{Q^2}{3\epsilon_0 m c^2} = \frac{4\pi}{3} r_0$$

This classical natural line width is independent of frequency
 $\sim 1.2 \times 10^{-4} \text{ \AA}$.

Q.Mech "oscillator strength" of $i \rightarrow j$ transition $f_{ij} = \Gamma_{\text{obs}} / \Gamma$

Note relation between $\langle P \rangle$ for the free oscillator and $\sigma(\omega)$ for the bound forced case.

$$\sigma(\omega) = \sigma_T \cdot \frac{\omega^4}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}$$

$$\begin{aligned} \text{For } \omega \approx \omega_0, \quad \omega_0^2 - \omega^2 &= (\omega_0 - \omega)(\omega_0 + \omega) \\ &\approx 2\omega_0(\omega_0 - \omega) \end{aligned}$$

$$\text{so } \sigma(\omega) = \sigma_T \cdot \frac{\omega_0^4}{4\omega_0^2(\omega - \omega_0)^2 + \omega_0^2 \Gamma^2}$$

$$\rightarrow \sigma_T \cdot \frac{\omega_0^2}{4(\omega - \omega_0)^2 + \Gamma^2}$$

$$\text{cf. } \langle P \rangle \propto \frac{1}{4(\omega - \omega_0)^2 + \Gamma^2} \quad (\text{same dependence on } \omega \text{ near } \omega_0)$$

Interpretation?

Free oscillation spectrum follows law of scattering from a forcing term that was white $e(\omega) = \text{constant}$

$$E(t) = \delta(t)$$

i.e. can be thought of as scattered spectrum of a sharp pulse.

10) Directivity of Scattering by free electrons

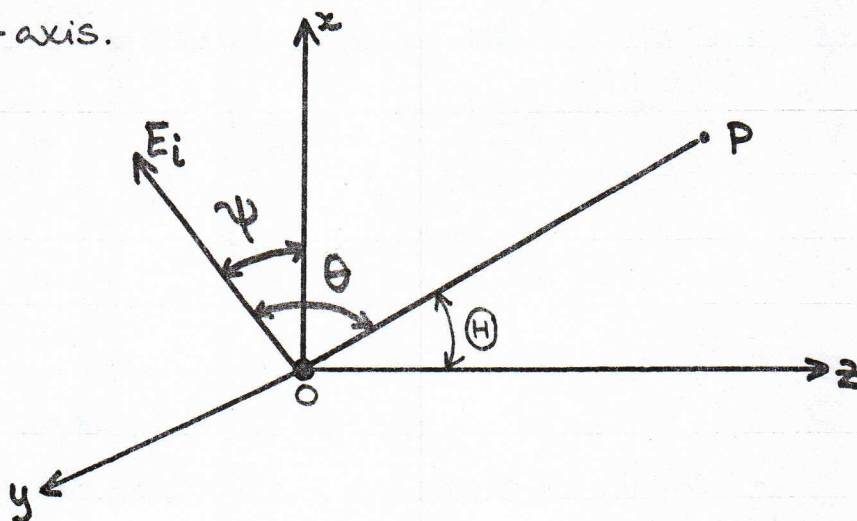
In treating only the total power scattered, some interesting aspects of Thomson scattering were glossed over above, namely the variation of the scattered power with direction and with wave polarisation.

To treat this, we go back to the expression for the power radiated into solid angle $d\Omega$ at an angle θ to the acceleration of the scattering charge:

$$\frac{dP(\theta)}{d\Omega} = \frac{Q^2 [\dot{v}]^2 \sin^2 \theta}{16\pi^2 \epsilon_0 c^3} = \frac{q^2 [\dot{\beta}]^2 \sin^2 \theta}{16\pi^2 \epsilon_0 c}$$

Suppose the incident wave is a linearly-polarised plane wave propagating in the z -direction. Define the field point to be in the (x,z) plane. Then in general the electric vector in the incident wave will be at an angle ψ to the x -axis.

θ is the angle between E_i (in the xy plane) and OP (in the xz plane). This is not the angle through which the radiation appears scattered, which is the angle Θ between OP and the original direction of propagation, i.e. the z -axis.



But note that projection of E_i onto OP could be made

- directly through angle θ
- first onto x -axis (through ψ), then onto OP (through $90 - \Theta$)
i.e. $\cos \theta = \cos \psi \sin \Theta$

Hence $\sin^2 \Theta = 1 - \cos^2 \Theta = 1 - \cos^2 \psi \sin^2 \Theta$.

and $\frac{dP(\Theta)}{d\Omega} = \frac{dP(\psi, \Theta)}{d\Omega} = Q^2 \cdot \left(\frac{QE}{m}\right)^2 \frac{1 - \cos^2 \psi \sin^2 \Theta}{16 \pi^2 \epsilon_0 c^3}$

$\frac{dP(\psi, \Theta)}{d\Omega \cdot \epsilon_0 c \langle E \rangle} = \frac{d\sigma(\psi, \Theta)}{d\Omega} = r_0^2 (1 - \cos^2 \psi \sin^2 \Theta)$ $(r_0^2 = \frac{q^2}{4\pi\epsilon_0 mc^2})$

This leads to an elegant interpretation of r_0^2 as the FORWARD ($\Theta = 0^\circ$) SCATTERING CROSS-SECTION for the free electron.

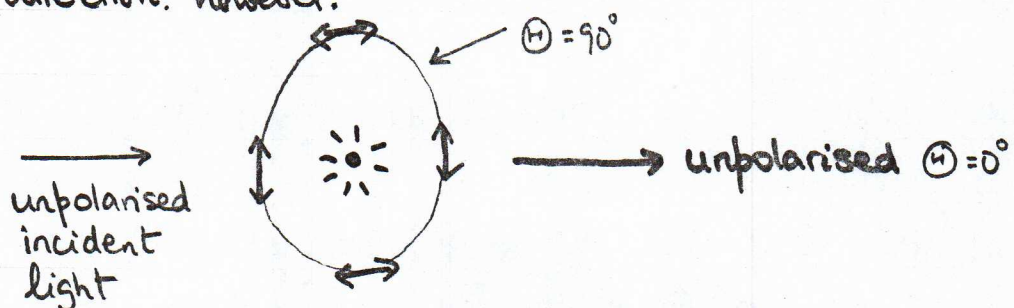
Note that there is no scattering from a linearly-polarised wave at right angles to the direction of propagation along the incident E-field.

If the incident radiation is "unpolarised", i.e. $\underline{E}$ varies randomly over all ψ , the average scattering cross-section is

$$\frac{d\bar{\sigma}(\Theta)}{d\Omega} = r_0^2 (1 - \overline{\cos^2 \psi} \sin^2 \Theta)$$

$$\frac{d\bar{\sigma}(\Theta)}{d\Omega} = r_0^2 \left(1 - \frac{1}{2} \sin^2 \Theta\right) = \frac{r_0^2 (1 + \cos^2 \Theta)}{2}$$

Note that in this case the forward scattering is twice the perpendicular scattering, which has the same intensity in all directions in the xy plane. The polarisation of the scattered radiation now varies with direction, however.



11) Radiation at High Velocities ($\beta \rightarrow 0$)

It is an elementary but tedious task to derive general high- v analogues for our low- v expressions for the directivity and total radiated power from an accelerated charge.

First note that the Poynting Vector $\underline{S}$ measures the power flux in the laboratory co-ordinates (x_f, t) where the fields are specified, and that

$$\underline{S} = \epsilon_0 c [E^2] [\underline{n}]$$

$$\text{so } [\underline{S} \cdot \underline{n}] = \frac{Q^2 [\underline{n} \times \{(\underline{n} - \underline{\beta}) \times \dot{\underline{\beta}}\}]^2}{16 \pi^2 \epsilon_0 c [R^2 K^6]}$$

measures the rate of energy flow across unit area of the observer's co-ordinates per unit time interval t .

We must be careful translating this into an expression for the ^{directional} energy loss rate of the accelerated charge however, for the energy crossing unit area per unit time t at the observer is not radiated in unit time interval t_- at the charge.

From the charge's point of view, the rate of energy loss into ^{unit} solid angle is:

$$- \frac{\partial \Sigma(\theta)}{\partial t_-} = [R^2 \underline{S} \cdot \underline{n}] \frac{dt}{dt_-} = [K R^2 (\underline{S} \cdot \underline{n})]$$

$$\text{i.e. } - \frac{\partial \Sigma(\theta)}{\partial t_-} = \frac{Q^2 [\underline{n} \times \{(\underline{n} - \underline{\beta}) \times \dot{\underline{\beta}}\}]^2}{16 \pi^2 \epsilon_0 c [K^5]}$$

At low velocities we could approximate $dt_- = dt$ ($K \rightarrow 1$).

To find the high-velocity generalisation of Larmor's formula for the total power radiated $P(t)$, or the total energy loss rate $-\partial\mathcal{E}/\partial t_-$ in all directions, we must integrate these expressions over all directions. When this is done, the effects of the different directivities of $dP(t)/d\Omega$ and of $-\partial\mathcal{E}(\theta)/\partial t_-$ cancel out, and we get what is known as Liénard's Formula:

$$P(t) = -\frac{\partial\mathcal{E}}{\partial t_-} = \frac{Q^2}{6\pi\epsilon_0 c} \gamma^6 [\dot{\beta}^2 - (\dot{\beta} \times \beta)^2]$$

Application of this to specific problems can still be very complicated, due to the geometrical problems in calculating the retarded bracket. Two limiting cases are straightforward, however.

1) Acceleration parallel to the velocity. $\dot{\beta} \times \beta = 0$

Here
$$P(t) = -\frac{\partial\mathcal{E}}{\partial t_-} = \frac{Q^2 \gamma^6 \dot{\beta}^2}{6\pi\epsilon_0 c}$$

i.e. radiation is γ^6 times more intense than in the low-velocity case for the same acceleration.

2) Acceleration perpendicular to the velocity $(\dot{\beta} \times \beta)^2 = \beta^2 \dot{\beta}^2$

Here
$$P(t) = -\frac{\partial\mathcal{E}}{\partial t_-} = \frac{Q^2 \gamma^6 \dot{\beta}^2}{6\pi\epsilon_0 c} (1 - \beta^2) = \frac{Q^2 \gamma^4 \dot{\beta}^2}{6\pi\epsilon_0 c}$$

i.e. radiation is γ^4 times more intense than in the low-velocity case for the same acceleration. This case is known as SYNCHROTRON RADIATION when the acceleration is provided by a static $\mathbf{B}$ field (analogous to the low-velocity cyclotron case).

It is worth noting that Liénard's Formula is easily derivable from Larmor's Formula using the techniques of Special Relativity.

If we apply Larmor's formula in the instantaneous rest-frame of the accelerated charge (coordinates with subscript 0)

$$-\frac{d\mathcal{E}_0}{dt_0} = \frac{Q^2}{6\pi\epsilon_0 c} \dot{\beta}_0^2$$

is the rate of energy loss into all directions in the charge's frame.

But by the Lorentz transformation for acceleration,

$$\dot{\beta}_0^2 = \gamma^6 (\dot{\beta}^2 - (\dot{\beta} \times \beta)^2)$$

where $\dot{\beta}$ is measured in an observer's frame in which the charge has instantaneous velocity factor β . Liénard's Formula follows trivially if we can show that $-d\mathcal{E}/dt_0$ also $= P(t) = -\partial\mathcal{E}/\partial t$

There are two factors to be considered. First, the energy lost in time interval Δt_0 by the charge is received in time interval Δt by the observer. The Lorentz transformation for the times gives us

$$\Delta t = \gamma \Delta t_0$$

i.e. Δt is longer by a factor γ .

But energy and momentum are computed differently in the two frames. The energy transformation is

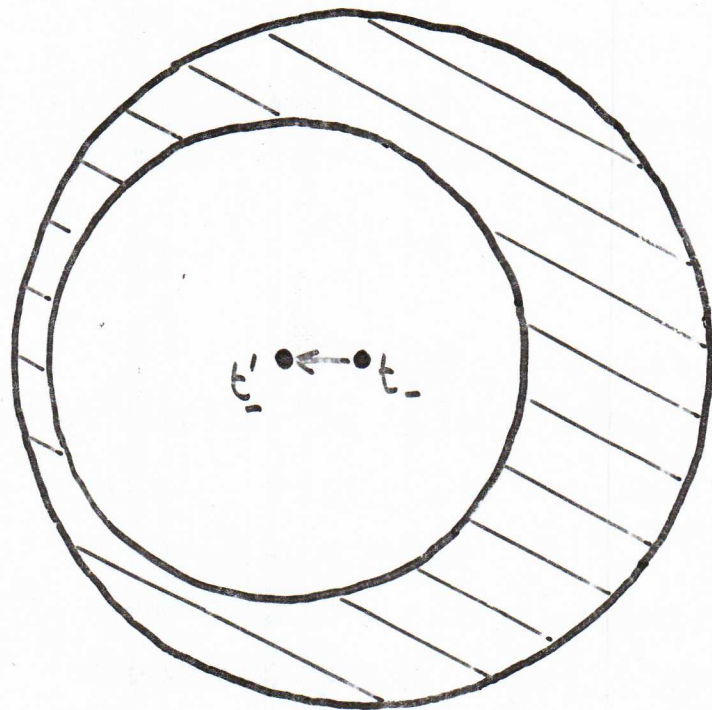
$$\mathcal{E} = \gamma (\mathcal{E}_0 - \underline{v} \cdot \underline{p}_0)$$

The total radiation in the charge's frame has the low-velocity $\sin^2\theta$ dependence, and so has no net momentum, $\underline{p}_0 = 0$. So its total energy in the observer's frame

$$\mathcal{E} = \gamma \mathcal{E}_0$$

Note that $\underline{p} = \gamma(\underline{p}_0 + \underline{v}E_0/c^2) \Rightarrow \gamma \underline{v}E_0/c^2$
(longitudinal)

This precisely cancels the time factor of γ , so $-\frac{\partial\mathcal{E}_0}{\partial t_0} = -\frac{\partial\mathcal{E}}{\partial t}$ as needed to complete our proof.

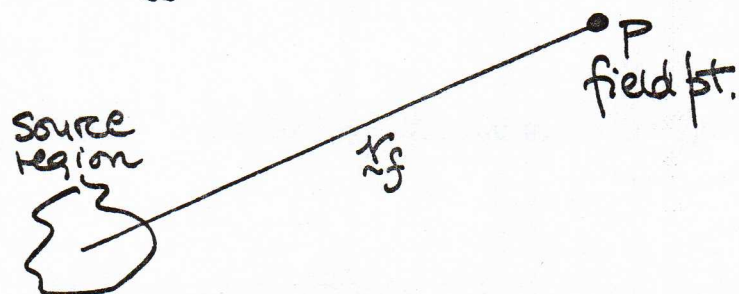


LOCATION OF ENERGY RADIATED BY MOVING CHARGE
($\beta \neq 0$) BETWEEN t_- AND t'_-

Radiation from particle ensembles

General system of many particles $\underline{r}_i, \underline{v}_i$, carrying q_i :

- in principle, just add $\underline{E}_{\text{rad}}$ from each particle, but practical complication: expressions for radiation fields refer to conditions at different retarded times for each particle.
- in practice, must keep track of phase relations between different parts of radiating system resulting from the different retardations.



For $|R| = |\underline{r}_f - \underline{r}_s| \gg \text{all } |\underline{r}_s|$
we can make some useful approximations :-

[As $\underline{E}$ and $\underline{B}$ are simply related, $\underline{A}$ contains all of the necessary information]

$\underline{A}$ is generally $\underline{A}(\underline{r}_f, t)$ produced by $\underline{j}(\underline{r}_s, t)$

Fourier analyse $\underline{J}(\underline{r}_s, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{j}(\underline{r}_s, t) e^{i\omega t} dt$

$$\underline{a}(\underline{r}_f, \omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{A}(\underline{r}_f, t) e^{i\omega t} dt$$

$$\text{and } \underline{A}(\underline{r}_f, t) = \frac{\mu_0}{4\pi} \iiint_{\text{all sources}} \frac{\underline{j}(\underline{r}_s, t_-)}{|\underline{R}|} dV_s$$

Specify $t_- = t - R/c$ by:

Using δ -fn, $\underline{A}(\underline{r}_f, t) = \frac{\mu_0}{4\pi} \iiint \frac{\underline{j}(\underline{r}_s, t_-)}{|\underline{R}|} \delta(t_- - t + \frac{R}{c}) dt_- dV_s$

Fourier-transforming this:-

$$a(\underline{r}_f, \omega) = \frac{\mu_0}{4\pi} \iiint \frac{1}{2\pi} \frac{j(\underline{r}_s, t_-)}{R} \delta(t_- - t + \frac{R}{c}) e^{i\omega t} dt dt_- dV_s$$

Now do the $\int dt$ FIRST, note that the δ -fn converts t to $(t_- + R/c)$

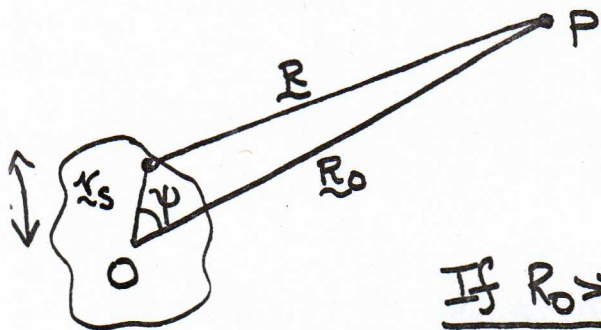
$$\begin{aligned} a(\underline{r}_f, \omega) &= \frac{\mu_0}{4\pi} \iiint \frac{1}{2\pi} \frac{j(\underline{r}_s, t_-)}{R} e^{i\omega(t_- + R/c)} dt_- dV_s \\ &= \frac{\mu_0}{4\pi} \iiint \frac{1}{2\pi} \frac{j(\underline{r}_s, t_-)}{R} e^{i\omega t_-} e^{ikR} dt_- dV_s \quad (k = \frac{\omega}{c} = \frac{2\pi}{\lambda}) \end{aligned}$$

Now do the $\int dt_-$, which is a Fourier transform integral

$$a(\underline{r}_f, \omega) = \frac{\mu_0}{4\pi} \iiint \frac{j(\underline{r}_s, \omega)}{R} e^{ikR} dV_s$$

Note interpretation
1/R amplitude
 e^{ikR} rel. phase.

Take origin of co-ordinates at a convenient point in the charge distribution, so that $OP \sim R_{\text{average}} = R_0$



$$\text{Exact } R^2 = r_s^2 + R_0^2 - 2r_s R_0 \cos \psi$$

$$R = R_0 \sqrt{1 - \frac{2r_s}{R_0} \cos \psi + \frac{r_s^2}{R_0^2}}$$

$$\text{If } R_0 \gg r_s, \quad R \rightarrow R_0 - r_s \cos \psi$$

$$\text{i.e. } R \rightarrow R_0 - (\underline{n} \cdot \underline{r}_s)$$

for P far from
source distrib

$$\text{So: } a(\underline{r}_f, \omega) \Rightarrow \frac{\mu_0}{4\pi} \iiint \frac{j(\underline{r}_s, \omega)}{R_0 - (\underline{n} \cdot \underline{r}_s)} e^{ikR_0} e^{-ik(\underline{n} \cdot \underline{r}_s)} dV_s$$

If $R_0 \gg L$, the largest scale of the source distribⁿ, variation of denominator across V_s is negligible, :-

$$a(\underline{r}_f, \omega) = \frac{\mu_0}{4\pi} \frac{e^{ikR_0}}{R_0} \iiint j(\underline{r}_s, \omega) e^{-ik(\underline{n} \cdot \underline{r}_s)} dV_s$$

Interpretation of terms:-

$e^{ikR_0} \rightarrow$ average retardation of source
 $e^{-ik(\underline{n} \cdot \underline{r}_s)} \rightarrow$ differential retardations
across source.

Now note that time-scale τ for significant change in source distribⁿ. must be $\tau = L/v$, where v is a typical charge velocity

This means that $\underline{J}(\underline{r}_s, \omega)$ is large only where $\omega < 2\pi/\tau$, i.e.

$$v < \frac{1}{\tau} = \frac{v}{L}$$

As $v = c/\lambda$ where λ is wavelength of fourier cpr ω in $\underline{J}$

$$\frac{c}{\lambda} < \frac{v}{L}, \text{ i.e. } \frac{L}{\lambda} < \frac{v}{c} \text{ at all } \lambda \text{ of interest.}$$

Therefore in the integrand $e^{-ik(\underline{n} \cdot \underline{r}_s)} < e^{-ikL} < e^{-i2\pi(v/c)}$

For non-relativistic motions ($v \ll c$) we can therefore expand the exponential as a power series

$$\underline{a}(\underline{r}_f, \omega) = \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} \iiint \underline{J}(\underline{r}_s, \omega) \sum_{N=0}^{\infty} \frac{1}{N!} (-ik\underline{n} \cdot \underline{r}_s)^N dV_s$$

This is the general multiple expansion for $\underline{a}(\underline{r}_f, \omega)$

First approx. Take only the first term $N=0$ — neglects the retardation differences across the source altogether.

[Valid for $v \ll c$, $L \ll \lambda$, $\tau \gg L/c$]

$$\underline{a}(\underline{r}_f, \omega) \Rightarrow \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} \iiint \underline{J}(\underline{r}_s, \omega) dV_s$$

Note that
$$\begin{aligned} \iiint \mathbf{J} dV &= \sum q_i \mathbf{v}_i \\ &= \sum q_i \dot{\mathbf{r}}_i \\ &= \dot{\mathbf{d}} \text{ where } \mathbf{d} \text{ is dipole moment of charge distribution} \end{aligned}$$

Hence $N=0$ is known as ^{ELECTRIC} DIPOLE TERM. This approximation is known as the ELECTRIC DIPOLE APPROXIMATION

Note that $\dot{\mathbf{d}}$ must still be evaluated at a retarded time, but this time is just that for one point in the source distribution (the one used to define R_0).

This equivalences the entire source distribution to a single particle oscillator at each ω , located "at" origin

$$\rightarrow \frac{dP(\theta)}{d\Omega} = \frac{\dot{\mathbf{d}}^2 \sin^2 \theta}{16\pi^2 \epsilon_0 c^3} = Z_0 \frac{\dot{\mathbf{d}}^2 \sin^2 \theta}{16\pi^2 c^2}$$

Note that these relations are restricted to non-relativistic particle motions

Second approx - include $N=0$ and $N=1$ terms

$$\mathcal{A}(\mathbf{r}_f, \omega) = \mathcal{A}_{\text{DIPOLE}}^{\text{EL}} + \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} \iiint \mathbf{J}(\mathbf{r}_s, \omega) (-ik \hat{\mathbf{n}} \cdot \mathbf{r}_s) dV_s$$

Consider
$$\iiint \mathbf{J}(\mathbf{r}, \tau) dV = \sum q_i \mathbf{v}_i(\mathbf{r}, \tau)$$

$$\frac{\partial}{\partial t} (\mathbf{r}_i(\mathbf{r}, \tau)) = \mathbf{v}_i(\mathbf{r}, \tau) + \mathbf{r}_i(\mathbf{r}, \tau)$$

$$\begin{aligned} \text{So } \mathbf{v}_i(\mathbf{r}, \tau) &= \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{r}_i(\mathbf{r}, \tau)) + \frac{1}{2} \mathbf{v}_i(\mathbf{r}, \tau) - \frac{1}{2} \mathbf{r}_i(\mathbf{r}, \tau) \\ &= \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{r}_i(\mathbf{r}, \tau)) + \frac{1}{2} (\mathbf{r}_i \times \mathbf{v}_i) \times \hat{\mathbf{n}} \end{aligned}$$

The second term involves the magnetic dipole moment $\underline{\mathcal{M}}$ of the current distribution

$$\begin{aligned}\underline{\mathcal{M}} &= \frac{1}{2} \sum_i q_i (\underline{r}_i \times \underline{v}_i) = \frac{1}{2} \iiint \rho (\underline{r}_s \times \underline{v}) dV_s \\ &= \frac{1}{2} \iiint (\underline{r}_s \times \underline{j}) dV_s\end{aligned}$$

The first term is related to the electric quadrupole moment $\underline{\underline{D}}$ of the charge distribution :-

$\underline{\underline{D}}$ is a tensor whose $\alpha\beta$ component is $D_{\alpha\beta} = \sum_i q_i (3x_{i\alpha}x_{i\beta} - r_i^2 \delta_{\alpha\beta})$

Now consider vector $\underline{D}$ constructed from $\underline{\underline{D}}$ and unit vector $\underline{n}$

α -cpt $D_\alpha = \sum_\beta n_\beta D_{\alpha\beta} = n_\beta D_{\alpha\beta}$ with summ. conv.

$$= \sum_i q_i (3x_{i\alpha} n_\beta x_{i\beta} - r_i^2 n_\beta \delta_{\alpha\beta})$$

i.e. $\underline{D} = \sum_i q_i (3 \underline{r}_i (\underline{n} \cdot \underline{r}_i) - r_i^2 \underline{n})$

and the first term in our expression is time derivative $\partial/\partial t$ of:-

$$\frac{1}{2} \sum_i q_i (\underline{r}_i (\underline{n} \cdot \underline{r}_i)) = \frac{1}{6} (\underline{D} + \sum_i q_i r_i^2 \underline{n})$$

Hence $\underline{a}(\underline{r}_f, \omega) = \underline{a}_{EL} + \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} \cdot (-ik) \left[\frac{1}{6} \frac{\partial}{\partial t} (\underline{D} + \sum_i q_i r_i^2 \underline{n}) + \underline{\mathcal{M}} \times \underline{n} \right]$

Now look at the term which is parallel to $\underline{n}$

$$\underline{a}_n = \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} \cdot (-ik) \cdot \frac{1}{6} \iiint \rho r_s^2 \underline{n} dV_s$$

THIS TERM IN THE VECTOR POTENTIAL DOES NOT CONTRIBUTE TO RADIATION FIELDS

To see why a_n does not contribute to radiation, consider

$$\underline{A} = A_r \hat{r} + A_\phi \hat{\phi} + A_\theta \hat{\theta} \quad (\text{sph. polar co-ords})$$

In our case $\hat{r} = \hat{r}_f \approx \underline{n}$ in the limit $R_0 \gg L$

Consider calculating $\underline{B}$ from $\nabla_f \times \underline{A}$ in the spherical polars:

$$= \frac{\hat{r}_f}{r_f \sin \theta} \left(\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right) + \frac{\hat{\theta}}{r_f} \left\{ \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r} (r_f A_\phi) \right\} + \frac{\hat{\phi}}{r_f} \left\{ \frac{\partial}{\partial r} (r_f A_\theta) - \frac{\partial A_r}{\partial \theta} \right\}$$

In this, consider the n-cpts of $\underline{A}$, i.e. the A_r . They contribute to $\underline{B}$ via:-

$$\frac{\hat{\theta}}{r_f} \left\{ \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} \right\} \quad \text{and} \quad - \frac{\hat{\phi}}{r_f} \frac{\partial A_r}{\partial \theta}$$

Now our A_r term (a_n) is $\propto \frac{e^{ikR_0}}{R_0}$, i.e. $\propto \frac{e^{ikr_f}}{r_f}$ Remember $R_0 = |r_f - \bar{r}_s|$

Hence both terms give $\underline{B}$ components $\propto \frac{1}{r_f^2} e^{ikr_f}$

i.e. a_n does not contribute RADIATIVE $\underline{B}$. Similarly for $\underline{E}$.

So the radiation fields are as if produced by (to order $N=1$)

$$\underline{Q}(r_f, \omega) = \underline{Q}_{\text{ELDIP}} + \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} \cdot (-ik) \cdot \left(\frac{1}{6} \frac{\partial \underline{D}}{\partial t} + \underline{M} \times \underline{n} \right)$$

ELECTRIC
QUADRUPOLE
TERM

MAGNETIC
DIPOLE
TERM

Note that the $N=1$ terms are of order $r_s k \sim r_s/\lambda$ times the $\underline{Q}_{\text{ELDIP}}$ term, so if the associated moments are similar, radiation fields via magnetic dipole or electric quadrupole are of order L/λ smaller than electric dipole, power smaller by $(L/\lambda)^2$.

Form of the fields:

Electric dipole: $\underline{B} = \underline{\nabla}_f \times \frac{\mu_0}{4\pi} \frac{e^{ikr_f}}{r_f} \dot{\underline{d}}$

Use: $\underline{\nabla} \times (f \underline{v}) = f(\underline{\nabla} \times \underline{v}) + \underline{\nabla} f \times \underline{v}$ f - scalar
v - vector

$$\underline{B} = \frac{\mu_0}{4\pi} \frac{e^{ikr_f}}{r_f} (\underline{\nabla}_f \times \dot{\underline{d}}) + \frac{\mu_0}{4\pi} \underline{\nabla}_f \left(\frac{e^{ikr_f}}{r_f} \right) \times \dot{\underline{d}}$$

Keeping only the terms in $1/r_f$ (radiation), the $\underline{\nabla}_f \rightarrow \frac{ik e^{ikr_f}}{r_f} \underline{n}$

$$\begin{aligned} \underline{B}_{\text{rad}} &= \frac{\mu_0}{4\pi} (ik) \frac{e^{ikr_f}}{r_f} (\underline{n} \times \dot{\underline{d}}) \\ &= \frac{\mu_0}{4\pi} (ik) \frac{e^{ikR_0}}{R_0} (\underline{n} \times \dot{\underline{d}}) \end{aligned}$$

Simplest way to find $\underline{E}_{\text{rad}}$ is to use $\underline{\nabla}_f \times \underline{B} = \mu_0 \epsilon_0 \dot{\underline{E}}$ (away from $\underline{J}$)

So for harmonic variation $\underline{\nabla}_f \times \underline{B} = \mu_0 \epsilon_0 \cdot (-i\omega) \underline{E}$
 $\propto e^{-i\omega t}$

$$\underline{E} = \left(\frac{c}{-ik} \right) \underline{\nabla}_f \times \underline{B}$$

$$\underline{E} = \left(\frac{c}{-ik} \right) \cdot \frac{\mu_0}{4\pi} (ik) \underline{\nabla}_f \times \left(\frac{e^{ikr_f}}{r_f} (\underline{n} \times \dot{\underline{d}}) \right)$$

$$\underline{E}_{\text{rad}} = - \frac{\mu_0 c}{4\pi} \cdot (ik) \frac{e^{ikr_f}}{r_f} \cdot (\underline{n} \times (\underline{n} \times \dot{\underline{d}}))$$

$$\underline{E}_{\text{rad}} = - \frac{ik}{4\pi \epsilon_0 c} \frac{e^{ikR_0}}{R_0} (\underline{n} \times (\underline{n} \times \dot{\underline{d}}))$$

$$\underline{B}_{\text{rad}} = \frac{ik \mu_0}{4\pi} \frac{e^{ikR_0}}{R_0} (\underline{n} \times \dot{\underline{d}})$$

Note we get back a version of fields for accelerated point charge

$P \propto \sin^2 \theta$ from these fields

cf. $\underline{E}_{\text{rad}} = \frac{Q}{4\pi \epsilon_0 R c} [\underline{n} \times (\underline{n} \times \dot{\underline{\beta}})]$ $\underline{B}_{\text{rad}} = \frac{Q}{4\pi \epsilon_0 R c^2} [-\underline{n} \times \dot{\underline{\beta}}]$

Magnetic dipole

$$\underline{\underline{B}} = \underline{\underline{\nabla}}_f \times \frac{\mu_0}{4\pi} \cdot \frac{e^{ikr_f}}{r_f} \cdot (-ik) \cdot (\underline{\underline{cM}} \times \underline{\underline{n}})$$

$$\begin{aligned} \underline{\underline{B}}_{\text{rad}} &= \frac{\mu_0}{4\pi} (-ik)(ik) \frac{e^{ikr_f}}{r_f} (\underline{\underline{n}} \times (\underline{\underline{cM}} \times \underline{\underline{n}})) \\ &= \frac{\mu_0}{4\pi} \cdot k^2 \cdot \frac{e^{ikR_0}}{R_0} (\underline{\underline{n}} \times (\underline{\underline{cM}} \times \underline{\underline{n}})) \end{aligned}$$

$$\begin{aligned} \underline{\underline{E}}_{\text{rad}} &= \left(\frac{c}{-ik} \right) \cdot \frac{\mu_0}{4\pi} \cdot k^2 \cdot (ik) \frac{e^{ikR_0}}{R_0} (\underline{\underline{n}} \times (\underline{\underline{n}} \times (\underline{\underline{cM}} \times \underline{\underline{n}}))) \\ &= + \frac{1}{4\pi\epsilon_0 c} \cdot k^2 \cdot \frac{e^{ikR_0}}{R_0} (\underline{\underline{cM}} \times \underline{\underline{n}}) \end{aligned}$$

Note that $\underline{\underline{E}}_{\text{rad}}$ for magn. dipole has same form as $\underline{\underline{B}}_{\text{rad}}$ for el. dipole
 $\underline{\underline{B}}_{\text{rad}} - - - - - \underline{\underline{E}}_{\text{rad}}$

but with $k\underline{\underline{cM}}$ replacing $\underline{\underline{d}}$, ($k\underline{\underline{M}}$ replacing $\underline{\underline{d}}$)

Hence magnetic dipole radiation is analogous in distribution to el. dipole

$$\frac{dP(\theta)}{d\Omega} = \frac{(k\underline{\underline{M}})^2 \sin^2 \theta}{16\pi^2 \epsilon_0 c^3}$$

$$\langle P \rangle = \frac{\dot{M}^2}{6\pi\epsilon_0 c^3} \cdot \frac{4\pi^2}{\lambda^2} \quad \text{is "Larmor" analog}$$

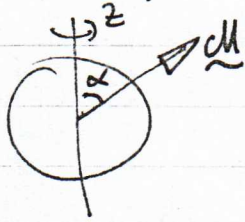
This radiation is produced by current distributions that have no net oscillating electric dipole moment, but does have sinusoidally-varying circulation of charges.

e.g. Pulsar, rotating neutron star (contains neutrons, protons & electrons)

Polar B-field, $B_0 = \frac{2cM}{R_*^3}$

If $\underline{M} \parallel$ to rotation axis $\underline{M}$ does not vary with time $\rightarrow$ no rad.

If $\underline{M}$ is at angle α to rotation axis (z)



$$M_z = M \cos \alpha$$

$$M_x = M \sin \alpha \sin \omega t$$

$$M_y = M \sin \alpha \cos \omega t$$

$$M = \frac{B_0 R_*^3}{2}$$

Analogy with el. dipole $k\underline{M}$ replaces $\underline{d}$
 $k\underline{\dot{M}}$ replaces $\underline{\dot{d}}$ in Larmor's formula.

$$\frac{dP(\theta)}{d\Omega} = \frac{\dot{d}^2 \sin^2 \theta}{16\pi^2 \epsilon_0 c^3} \Rightarrow \frac{k^2 \dot{M}^2 \sin^2 \theta}{16\pi^2 \epsilon_0 c^3} \Rightarrow \frac{k^2 \omega^2 B_0^2 R_*^6 \sin^2 \theta \sin^2 \alpha}{64\pi^2 \epsilon_0 c^3}$$

$$\Rightarrow \frac{\omega^4 B_0^2 R_*^6 \sin^2 \theta \sin^2 \alpha}{64\pi^2 \epsilon_0 c^5}$$

$$P \Rightarrow \frac{\omega^4 B_0^2 R_*^6 \sin^2 \alpha}{24\pi \epsilon_0 c^5}$$

Then $E_{\text{rot}} \sim \frac{2}{5} M_* R_*^2 \omega^2$

$$P = -\dot{E}_{\text{rot}} = -\frac{4}{5} M_* R_*^2 \omega \dot{\omega} = \frac{\omega^4 B_0^2 R_*^6 \sin^2 \alpha}{24\pi \epsilon_0 c^5}$$

Spin-down time $\tau \sim \frac{\omega}{\dot{\omega}} \sim \left(\frac{5\omega^2 B_0^2 R_*^4 \sin^2 \alpha}{96\pi \epsilon_0 M_* c^5} \right)^{-1}$

$M_* \sim 1 M_\odot$
 $R_* \sim 10^6 \text{ cm}$
 $B_0 \sim 10^{12} \text{ gauss}$

Fastest $\omega \sim 10^4$
 $\tau \sim \frac{84 \text{ yrs}}{\sin^2 \alpha}$

$$\sim \frac{96\pi G_0 M_* c^5}{5\omega^2 B_0^2 R_*^4 \sin^2 \alpha}$$

Electric quadrupole

$$\underline{\underline{B}} = \underline{\underline{\nabla}}_f \times \frac{\mu_0}{4\pi} \cdot \frac{e^{ikr_f}}{r_f} \cdot (-ik) \cdot \frac{1}{6} \underline{\underline{D}}$$

Here by comparison with el. dipole $-\frac{ik\underline{\underline{D}}}{6} \leftrightarrow \underline{\underline{d}}$

$$\underline{\underline{E}}_{\text{rad}} \rightarrow -\frac{k^2}{4\pi\epsilon_0 c} \cdot \frac{1}{6} \cdot \frac{e^{ikR_0}}{R_0} (\underline{\underline{n}} \times (\underline{\underline{n}} \times \underline{\underline{D}}))$$

$$\underline{\underline{B}}_{\text{rad}} \rightarrow \frac{k^2 \mu_0}{4\pi} \cdot \frac{1}{6} \cdot \frac{e^{ikR_0}}{R_0} (\underline{\underline{n}} \times \underline{\underline{D}})$$

In general the $\underline{\underline{D}}$ has a very complex angular distribution, so this directionality is not as simple as it looks!

$$\left\langle \frac{dP(\theta)}{d\Omega} \right\rangle = \frac{1}{2} \mu_0 c |H|^2 = \frac{1}{2} \mu_0 c \frac{k^4}{576\pi^2} \cdot k^2 c^2 |\underline{\underline{n}} \times \underline{\underline{D}}|^2$$

$$= \frac{ck^6}{1152\pi^2 \epsilon_0} |\underline{\underline{n}} \times \underline{\underline{D}}|^2$$

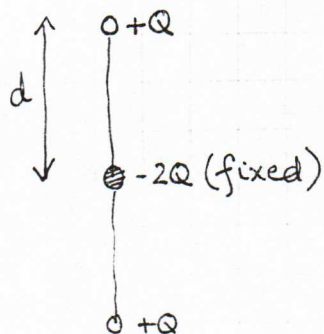
for harmonic
at ω

Now for quadrupole $H_{\text{rad}} = \frac{k^2}{24\pi} \cdot \frac{e^{ikR_0}}{R_0} (\underline{n} \times \underline{\dot{D}})$

$$\langle (\underline{S} \cdot \underline{n}) \rangle_{R_0}^2 = \left\langle \frac{dP(\theta)}{d\Omega} \right\rangle = \frac{1}{2} \mu_0 c |H|^2 = \frac{1}{2} \mu_0 c \cdot \frac{k^4}{576\pi^2} \cdot \frac{k^2 c^2}{\cancel{}} |\underline{n} \times \underline{\dot{D}}|^2$$

$$\boxed{\left\langle \frac{dP(\theta)}{d\Omega} \right\rangle = \frac{c k^6 |\underline{n} \times \underline{\dot{D}}|^2}{1152\pi^2 \epsilon_0}}$$

Simple quadrupole system (an example)



⊕ charges oscillate harmonically ^{at freq ω} with amplitude d along z -axis

→ no DIPOLE moments.

$$D_{\alpha\beta} = \sum_i (3x_\alpha x_\beta - r_i^2 \delta_{\alpha\beta}) Q_i$$

For $\alpha \neq \beta$, all $\rightarrow 0$ because $x=0, y=0$ always

$$\alpha = \beta = 1 \quad |Q_{11}| = -d^2 Q_1 - d^2 Q = -2d^2 Q$$

$$\alpha = \beta = 2 \quad |Q_{22}| = -d^2 Q - d^2 Q = -2d^2 Q$$

$$\alpha = \beta = 3 \quad |Q_{33}| = 3d^2 Q - d^2 Q + 3d^2 Q - d^2 Q = 4d^2 Q$$

$$\text{So } \underline{\underline{D}} = 2Q \begin{pmatrix} -d^2 & 0 & 0 \\ 0 & -d^2 & 0 \\ 0 & 0 & 2d^2 \end{pmatrix} e^{i\omega t}$$

$$\underline{D} = \underline{\underline{D}} \underline{n} = (-2Qd^2 n_x \underline{i} - 2Qd^2 n_y \underline{j} + 4Qd^2 n_z \underline{k}) e^{i\omega t}$$

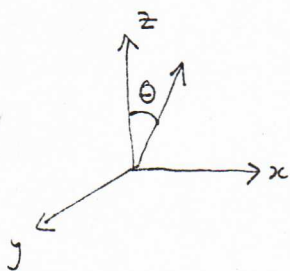
$$\text{Hence } |\underline{n} \times \underline{D}| = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ n_x & n_y & n_z \\ -2Qd^2 n_x & -2Qd^2 n_y & 4Qd^2 n_z \end{vmatrix} = 6Qd^2 n_y n_z \underline{i} + 6Qd^2 n_x n_z \underline{j} + 0$$

$$|\underline{n} \times \underline{D}|^2 = (6Qd^2)^2 (n_y^2 n_z^2 + n_x^2 n_z^2)$$

$$= 36Q^2 d^4 n_z^2 (n_x^2 + n_y^2)$$

$$= 36Q^2 d^4 n_z^2 (1 - n_z^2)$$

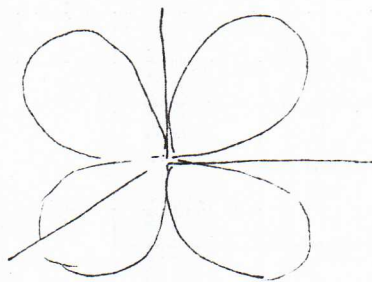
$$\text{So: } |\underline{n} \times \underline{D}|^2 = 36Q^2 d^4 \sin^2 \theta \cos^2 \theta$$



Hence for this system $\left\langle \frac{dP(\theta)}{d\Omega} \right\rangle = \frac{ck^6 Q^2 d^4 \sin^2 \theta \cos^2 \theta}{32\pi^2 \epsilon_0}$

Polar diagram:-

$\sim \sin^2 2\theta$



(Two nodal lines are characteristic of quadrupole distributions).

Total radiated power is $\langle P \rangle = \frac{ck^6 Q^2 d^4}{32\pi^2 \epsilon_0} \cdot \frac{8\pi}{15}$

i.e. $\langle P \rangle = \frac{ck^6 Q^2 d^4}{60\pi \epsilon_0}$

Compare this with the electric dipole

$\uparrow$ $\circ - Q$ (fixed)

$\downarrow$ $\circ + Q$ (moving harmonically with amplitude d)

By Larmor, $\langle P \rangle = \frac{\langle \ddot{d}^2 \rangle}{6\pi\epsilon_0 c^3} = \frac{\omega^4 Q^2 d^2}{12\pi\epsilon_0 c^3} = \frac{k^4 c^4 Q^2 d^2}{12\pi\epsilon_0 c^3}$
 $= \frac{k^4 c Q^2 d^2}{12\pi\epsilon_0}$

Hence $\frac{\langle P \rangle_{\text{quad}}}{\langle P \rangle_{\text{dip}}} = \frac{ck^6 Q^2 d^4}{60\pi\epsilon_0} \cdot \frac{12\pi\epsilon_0}{ck^4 Q^2 d^2}$
 $= \frac{1}{5} k^2 d^2$

Note, for systems of identical particles $\rho(r_s) = g \mu(r_s)$
 where g is the charge-to-mass ratio and μ is the mass density

Therefore the electric dipole term

$$\begin{aligned} \underline{a}(\underline{r}_f, \omega) &= \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} \iiint \underline{J}(\underline{r}_s, \omega) dV_s \\ &= \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} \iiint g \underline{v}(\underline{r}_s, \omega) dV_s \\ &= \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} g \iiint \mu \underline{v}(\underline{r}_s, \omega) dV_s \\ &= \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} g \iiint \underline{p}(\underline{r}_s, \omega) dV_s \\ &= \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} g \underline{P}(\omega) \quad \text{where } \underline{P} \text{ is the linear momentum} \end{aligned}$$

For an isolated system, $\underline{P}(t) = \text{constant}$, $\underline{P}(\omega) = 0$ for all $\omega > 0$.
 $\rightarrow$ NO ELECTRIC DIPOLE RADIATION!

Similarly, the magnetic dipole term for systems of identical particles

$$\begin{aligned} \underline{a}(\underline{r}_f, \omega) &= \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} (-ik) \iiint \frac{1}{2} (\underline{r}_s \times g \underline{v}) \times \underline{n} dV_s \\ &= \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} (-ik) \frac{g}{2} \left[\iiint (\underline{r}_s \times \underline{p}) dV_s \right] \times \underline{n} \\ &= \frac{\mu_0}{4\pi} \cdot \frac{e^{ikR_0}}{R_0} (-ik) \frac{g}{2} \left[\underline{L} \times \underline{n} \right] \end{aligned}$$

$\underline{L}$ is total angular momentum.

$\underline{L}(t) = \text{constant}$ for isolated system, so NO MAGN. DIPOLE RADN.
 HIGHEST TERM IS EL. QUADRUPOLE.

12) Radiation from Antennas (Jackson, Ch. 9; Marion, Ch. 8; Papas, Ch. 3)

General antenna-radiation problem could be solved by going back to accelerated-charge fields. However, some constraints unique to antenna problems affect language of solutions.

- 1) Sources are harmonic functions of time $\rightarrow \underline{A}(\underline{r}_f, t) = \underline{a}(\underline{r}_f, \omega) e^{-i\omega t}$ at one ω only.
- 2) But, $\lambda \gg L$ not generally true, indeed $L > \lambda$ needed.
- 3) Geometry of sources is however fixed in time.
- 4) Normally are interested in $R_0 \gg L$ — radiation fields in far field of antenna.

Hence return to our previous discussion just before taking the multipole expansion

$$\underline{a}(\underline{r}_f) = \frac{\mu_0}{4\pi} \frac{e^{ikR_0}}{R_0} \iiint \underline{J}(\underline{r}_s) e^{-ik(\underline{n} \cdot \underline{r}_s)} dV_s$$

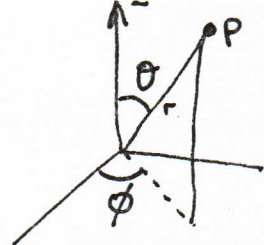
and $\underline{A} = \underline{a} e^{-i\omega t}$

$$\begin{aligned} \text{so } \underline{A}(\underline{r}_f) &= \frac{\mu_0}{4\pi} e^{-i\omega t} \frac{e^{ikR_0}}{R_0} \iiint \underline{J}(\underline{r}_s) e^{-ik(\underline{n} \cdot \underline{r}_s)} dV_s \\ &= f \underline{N} \end{aligned}$$

The factor f is obviously common to all antenna calculations.
The vector $\underline{N}$ depends on antenna excitation and geometry.

Its central rôle in determining radiative properties of antennas was first set out systematically by Schelkunoff (Proc. I.R.E., 27, 660 (1939)). It is known as SCHELKUNOFF'S RADIATION VECTOR

Now consider $\underline{N} = N_r \hat{r} + N_\theta \hat{\theta} + N_\phi \hat{\phi}$



$$\underline{B} = \nabla \times \underline{A}$$

Hence, e.g. $B_r = \frac{1}{r_f \sin \theta} \left\{ \frac{\partial}{\partial \theta} (f N_\phi \sin \theta) - \frac{\partial}{\partial \phi} (f N_\theta) \right\}$

f does not depend on θ or ϕ , so it can be taken out

$$B_r = \frac{f}{r_f \sin \theta} \left\{ \frac{\partial N_\phi}{\partial \theta} \sin \theta + N_\phi \cos \theta - \frac{\partial N_\theta}{\partial \phi} \right\}$$

Now N is a function of r_s only, not of r_f . So the factor outside the bracket contains the entire dependence on r_f

$$\frac{f}{r_f} = \frac{\mu_0}{4\pi} e^{-i\omega t} \frac{e^{ik(r_f - r_s)}}{r_f |r_f - r_s|} \sim \frac{1}{r_f^2} \quad (r_f \gg r_s)$$

i.e. these B_r components are not radiation fields. $\therefore$ forget them

$$\begin{aligned} \text{now } B_\theta &= \frac{1}{r_f} \left\{ \frac{1}{\sin \theta} \frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r_f} (r_f A_\phi) \right\} \\ &= \frac{1}{r_f} \left\{ \frac{f}{\sin \theta} \cdot \frac{\partial N_r}{\partial \phi} - \frac{\partial}{\partial r_f} \left(\frac{\mu_0}{4\pi} e^{-i\omega t} e^{ik(r_f - r_s)} \overbrace{N_\phi}^{\text{far from source}} \right) \right\} \end{aligned}$$

First term is again not a radiation field ($\propto 1/r_f^2$)

Second term is $-\frac{1}{r_f} \frac{\partial}{\partial r_f} \left(\frac{\mu_0}{4\pi} e^{-i\omega t} e^{ikr_f} e^{-ikr_s} N_\phi \right)$

Take out terms that don't depend on r_f : $-\frac{1}{r_f} \cdot \frac{\mu_0}{4\pi} e^{-ikr_s} N_\phi (ike^{ikr_f}) = -ikfN_\phi$
i.e. $B_\theta = -ikfN_\phi$

Similarly, only radiation term in B_ϕ is $+ikfN_\theta$

$$B_\phi = ikfN_\theta$$

Now $\underline{E} = -\underline{\nabla}\phi - \underline{\dot{A}}$

and from the gauge, $\underline{\nabla} \cdot \underline{A} = -\frac{1}{c^2} \dot{\phi}$ $\dot{\phi} = -c^2 \underline{\nabla} \cdot \underline{A}$

For $e^{-i\omega t}$ harmonic time dependence, $\dot{\phi} = -i\omega\phi$ $\dot{\underline{A}} = -i\omega \underline{A}$
 so $\phi = \frac{c^2}{\omega} \underline{\nabla} \cdot \underline{A}$

so $\underline{E} = -\frac{c^2}{\omega} \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) + i\omega \underline{A}$

Substituting $\underline{A} = f \underline{N}$ and arguing as before, only surviving terms in $1/r_f$ are

$$\begin{cases} E_\theta = i\omega f N_\theta \\ E_\phi = i\omega f N_\phi \end{cases}$$

[N.B. the E_r term from $i\omega \underline{A} = i\omega f \underline{N}$ is cancelled by a term in $\underline{\nabla}(\underline{\nabla} \cdot \underline{A})$]

Hence the time-averaged Poynting Vector is $\epsilon_0 c \overline{E^2} = \frac{1}{2} \epsilon_0 c E_{\max}^2$

$$\frac{d\langle P \rangle(\theta, \phi)}{d\Omega} = \frac{Z_0}{8\lambda^2} \{ |N_\theta|^2 + |N_\phi|^2 \}$$

Note $Z_0 = \mu_0 c = 120\pi \text{ ohms}$

$$\langle P \rangle = \frac{Z_0}{8\lambda^2} \int_0^{2\pi} \int_0^\pi \{ |N_\theta|^2 + |N_\phi|^2 \} \sin\theta d\theta d\phi$$

(a) Antennas in which all currents are parallel.

Take z-axis parallel to the currents

Then clearly $\underline{N}$ is in the z-direction, so

$$\begin{cases} \underline{N} = N \hat{z} \\ N_\phi = 0 \\ N_\theta = -N \sin\theta \end{cases}$$

θ : def. of $\hat{z}$

i.e. $\frac{d\langle P \rangle(\theta)}{d\Omega} = \frac{Z_0 N^2 \sin^2\theta}{8\lambda^2}$

note dipole-like $\sin^2\theta$ will be modified by $N(\theta)$

$$\text{and } \langle P \rangle = \frac{Z_0}{8\lambda^2} \int_0^{2\pi} \int_0^\pi N^2 \sin^3\theta d\theta d\phi$$

Short antenna ($l \ll \lambda$). Excitation $I_0 e^{-i\omega t}$ (electric dipole case)

For the short current element of length $l \ll \lambda$, $N \Rightarrow \int_0^l I_0 \hat{z} e^{-ikz \cos \theta} dz$
 $(N = \frac{I_0 l \hat{z}}{I_0 l})$ ($e^{ikz \cos \theta} \rightarrow 1$)

So: $\frac{d\langle P \rangle}{d\Omega} = \frac{Z_0 I_0^2}{8} \left(\frac{l}{\lambda}\right)^2 \sin^2 \theta$ dipolar directivity

$$\langle P \rangle = \frac{Z_0 I_0^2}{8} \left(\frac{l}{\lambda}\right)^2 \int_0^{2\pi} d\phi \int_0^\pi \sin^3 \theta d\theta$$

$$\text{i.e. } \langle P \rangle = \frac{\pi Z_0 I_0^2}{3} \left(\frac{l}{\lambda}\right)^2$$

Treating the antenna as a circuit element, we define the radiation resistance R_r by $\langle P \rangle = R_r \cdot I_0^2 / 2$

$$\text{i.e. } R_r = \frac{2\pi Z_0}{3} \left(\frac{l}{\lambda}\right)^2$$

$$R_r = 80\pi^2 \left(\frac{l}{\lambda}\right)^2 \text{ ohms.}$$

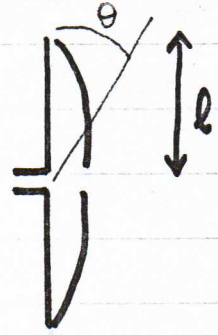
Maintaining the approximation $l \ll \lambda$, we can only expect $R_{rad} \sim$ a few ohms. For efficiency we will need $R_{rad} \gg R_{ohmic}$ in a real antenna. So this low radiation resistance is undesirable.

In practice, the $\sin^2 \theta$ directionality is also undesirable for many purposes. The HPBW is the angular distance between directions for which $d\langle P \rangle / d\Omega = \frac{1}{2} (d\langle P \rangle / d\Omega)_{max}$. As $\sin \theta = 1/\sqrt{2}$ for $\theta = 45^\circ$, the HPBW of the short dipole is 90° . Another convenient measure is the Directivity $= (d\langle P \rangle / d\Omega)_{max} / (\langle P \rangle / 4\pi) = \text{Max Power} / \text{Av. Power}$

$$\text{For the short dipole } D = \frac{\frac{Z_0 I_0^2}{8} (l/\lambda)^2}{(d\langle P \rangle / d\Omega)_{max}} \cdot 4\pi \cdot \frac{3}{\pi Z_0 I_0^2 (l/\lambda)^2} = 1.5$$

Real linear thin antenna with sinusoidal excitation

A more useful antenna is the centre-fed thin antenna. Actual excitations are quasi-sinusoidal, so if the gap is negligibly small we can put the excitation:



$$I(z) = \begin{cases} I_0 \sin(k(l-z)) & z > 0 \\ I_0 \sin(k(l+z)) & z < 0 \end{cases}$$

CANNOT APPROX.

Now the Schelkunoff Vector is:

$$\underline{N} = I_0 \hat{z} \left\{ \int_{-l}^0 \sin(k(l+z)) e^{-ikz \cos \theta} dz + \int_0^l \sin(k(l-z)) e^{-ikz \cos \theta} dz \right\}$$

$$= \frac{2I_0 \hat{z}}{k \sin^2 \theta} \left\{ \cos(kl \cos \theta) - \cos kl \right\} \quad \text{after carrying out the integrations}$$

$$\left[\int e^{ax} \sin(bx+c) dx = \frac{e^{ax}}{a^2+b^2} [a \sin(bx+c) - b \cos(bx+c)] \right]$$

$$\text{So } \frac{d\langle P \rangle}{d\Omega} = \frac{Z_0 I_0^2}{8\pi^2 \sin^2 \theta} \left\{ \cos(kl \cos \theta) - \cos kl \right\}^2$$

$$\text{and } \langle P \rangle = \frac{Z_0 I_0^2}{4\pi} \int_0^\pi \frac{\left\{ \cos(kl \cos \theta) - \cos kl \right\}^2}{\sin \theta} d\theta$$

$$\text{This gives } R_r = 60 \left\{ C + \ln(2kl) - \text{Ci}(2kl) + \frac{1}{2} \sin 2kl [\text{Si}(4kl) - 2\text{Si}(2kl)] + \frac{1}{2} \cos 2kl [C + \ln(kl) + \text{Ci}(4kl) - 2\text{Ci}(2kl)] \right\} \text{ ohms}$$

$$\text{where } C = 0.5772$$

$$\text{Si}(x) = \int_0^x \frac{\sin x}{x} dx$$

$$\text{and } \text{Ci}(x) = - \int_x^\infty \frac{\cos x}{x} dx$$

The general case is thus algebraically awful, but some practical cases are simple resonant forms, as follows:

Half-wave antenna $l = \lambda/4$

$$\frac{d\langle P \rangle}{d\Omega} \rightarrow \frac{15 I_0^2}{\pi} \left\{ \frac{\cos(\frac{\pi}{2} \cos \theta)}{\sin \theta} \right\}^2$$



$$R_r \rightarrow 73.09 \text{ ohms}$$

$$\text{HPBW} = 78^\circ \quad \text{Directivity } D = 1.64$$

Full-wave antenna $l = \lambda/2$

$$\frac{d\langle P \rangle}{d\Omega} \Rightarrow \frac{60 I_0^2}{\pi} \left\{ \frac{\cos^2(\frac{\pi}{2} \cos \theta)}{\sin^2 \theta} \right\}$$



N.B. IN
PHASE ON
OPP. SIDES

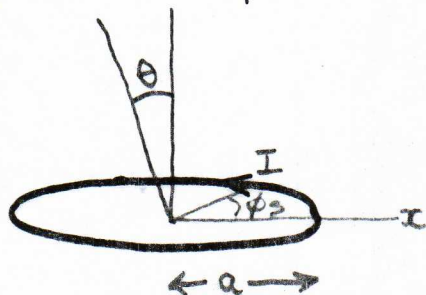
$$R_r \rightarrow 201 \text{ ohms}$$

$$\text{HPBW} = 48^\circ \quad \text{Directivity } D = 2.41$$

Note that increasing the length of the collinear antenna has more effect on R_r than on D . To get very directive antennas you are forced to apertures $\gg \lambda$, which are best excited by continuous-aperture techniques rather than by wires (to be discussed later).

b) Antennas on which currents are circularly symmetric

e.g. circular loop antenna.



Consider one of small radius $a \ll \lambda$, carrying a spatially constant current $I e^{i\omega t}$

Field point direction θ, ϕ } ϕ 's measured from x towards y (as usual)
Source point azimuth ϕ_s

Note that in general $N_\theta = (N_x \cos\phi + N_y \sin\phi) \cos\theta - N_z \sin\theta$
 $N_\phi = -N_x \sin\phi + N_y \cos\phi$

Because of the circular symmetry, N_ϕ and N_θ must be independent of ϕ
 So can compute them at $\phi = 0$ without loss of generality, $N_\theta = N_x \cos\theta$, $N_\phi = N_y$ there

In this problem $N_z = 0$ as currents are all in the (x, y) plane

$$\cos\psi = \sin\theta \cos\phi_s$$

$$\hat{\phi} \equiv \hat{y} \text{ at } \phi = 0.$$

$$\text{Now } \underline{N} = \iiint \underline{J}(\underline{r}_s) e^{-ikr_s \cos\psi} dV_s \rightarrow \underline{N} = \oint \underline{I} e^{-ik a \sin\theta \cos\phi_s} dl$$

$$\left. \begin{aligned} \text{so } N_x &= \oint I \sin\phi_s e^{-ik a \sin\theta \cos\phi_s} a d\phi_s \\ N_y &= \oint I \cos\phi_s e^{-ik a \sin\theta \cos\phi_s} a d\phi_s \end{aligned} \right\} \text{ are the integrals to be evaluated}$$

For $a \ll \lambda$, $e^{-ik a \sin\theta \cos\phi_s} \sim 1 - ik a \sin\theta \cos\phi_s$

As the integrals over ϕ_s run from 0 to 2π ,

$$N_x = \int_0^{2\pi} (I a \sin\phi_s - ik I a^2 \sin\theta \sin\phi_s \cos\phi_s) d\phi_s = 0$$

$$N_y = \int_0^{2\pi} (I a \cos\phi_s - ik I a^2 \sin\theta \cos^2\phi_s) d\phi_s = -ik\pi I a^2 \sin\theta$$

So we have $|N_\theta| = 0$ and $|N_\phi| = ik\pi I a^2 \sin\theta$ (N_θ always = 0 for this geometry)

$$\begin{aligned} \text{Hence } \frac{d\langle P \rangle}{d\Omega} &\rightarrow \frac{120\pi}{8\lambda^2} \cdot k^2 \pi^2 I^2 a^4 \sin^2\theta \\ &= \frac{15\pi}{4} (ka)^4 I^2 \sin^2\theta \end{aligned}$$

$$\text{and } \langle P \rangle = 10\pi^2 (ka)^4 I^2$$

$$\text{so } R_{\text{rad}} = 20\pi^2 (ka)^4 \text{ ohms}$$

This circular-loop antenna has the same directionality ($\sin^2\theta$) as the short linear antenna and is also described as a DIPOLE antenna for this reason.

Its efficiency as a radiator is much less however:

$$\text{Linear "Dipole" (Electric)} \quad R_{\text{rad}} = 80\pi^2 (\ell/\lambda)^2 \text{ ohms}$$

$$\text{Circular "Dipole" (Magnetic)} \quad R_{\text{rad}} = 20\pi^2 (ka)^4 \text{ ohms}$$

To get high directivity, there is no way other than to make dimensions $\gg \lambda$.

The problem of calculating the radiation pattern then becomes a problem in diffraction through an extended aperture defined by the antenna boundaries.

The analysis of large antennas is therefore paralleled by the theory of propagation through apertures, and is part of the general topic of DIFFRACTION

Theory of Diffraction

13) Fields as Sources of Radiation - Kirchhoff's Formulation

(Jackson, Ch. 9; Marion, Ch. 12)

Consider an anharmonic $\underline{E}, \underline{B}$ field $\underline{E}(\underline{r}, t), \underline{B}(\underline{r}, t)$ in free space away from sources.

The Fourier representations $\underline{e}(\underline{r}, \omega), \underline{b}(\underline{r}, \omega)$ satisfy $\underline{e}(\underline{r}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{E}(\underline{r}, t) e^{i\omega t} dt$
etc.

The representations satisfy $\underline{\nabla} \cdot \underline{b} = 0$

$$\underline{\nabla} \cdot \underline{e} = 0$$

$$\left. \begin{aligned} \underline{e}/c &= \underline{\nabla} \times \underline{b} = -i\omega \underline{e}/c^2 \\ -\underline{b} &= \underline{\nabla} \times \underline{e} = i\omega \underline{b} \end{aligned} \right\} \text{ recall that } \underline{E}(\underline{r}) = \int_{-\infty}^{\infty} \underline{e}(\omega) e^{-i\omega t} d\omega$$

$$\left. \begin{aligned} \text{i.e. } \underline{\nabla} \times (\underline{\nabla} \times \underline{b}) &= -i\omega (\underline{\nabla} \times \underline{e})/c^2 = \omega^2 \underline{b}/c^2 \\ &= \underline{\nabla} (\underline{\nabla} \cdot \underline{b}) - \nabla^2 \underline{b} = -\nabla^2 \underline{b} \end{aligned} \right\} \text{ i.e. } (\nabla^2 + k^2) \underline{b} = 0$$

Similarly, $(\nabla^2 + k^2)\underline{e} = 0$

i.e. away from their sources, the field spectra $e(\omega)$, $b(\omega)$ obey the HOMOGENEOUS VECTOR HELMHOLTZ EQUATION.

i.e. the field distributions in a distant region can be related to those in an aperture by solution of $(\nabla^2 + k^2)\underline{F} = 0$ once the aperture fields produced by actual sources are known. This forms the basis of the description of fields as sources.

The exact treatment is vectorial. We must solve the Helmholtz Equation for $\underline{e}$, $\underline{b}$, hence find $\underline{E} \times \underline{H}$. The boundary conditions on $\underline{e}$ and $\underline{b}$ will be imposed by real physical boundaries at the edges of apertures, i.e. they will be vectorial conditions with parallel and perpendicular components.

Fortunately, in many applications we don't need a fully vectorial theory.

- 1) we want power flow, $|S| \propto |e|^2$ or $|b|^2$
- 2) we deal with apertures $\gg \lambda$ and apertures whose areas $\gg$ perimeters $\times$ transition depths

i.e. we can construct a theory based on SCALAR equation

$$(\nabla^2 + k^2)\psi = 0$$

in which the wave power will be taken $\propto \psi^2$

and boundary conditions will be $\psi \rightarrow 0$ over infinitesimal depths.

Evidently a scalar theory cannot treat such vectorial phenomena as wave polarization.

The scalar theory is based on application of the Divergence Theorem to combinations of solutions of the scalar Helmholtz Equation, i.e. through the application of Green's Theorem:

Preliminary: construct a vector $\underline{v} = \psi_1 \nabla \psi_2$ and write the Divergence Theorem:

$$\oint_S \underline{v} \cdot d\underline{S} = \iiint_V (\nabla \cdot \underline{v}) dV$$

$$\oint_S \psi_1 (\nabla \psi_2 \cdot d\underline{S}) = \iiint_V \{ \psi_1 \nabla^2 \psi_2 + (\nabla \psi_1 \cdot \nabla \psi_2) \} dV$$

so $\oint_S \psi_2 (\nabla \psi_1 \cdot d\underline{S}) = \iiint_V \{ \psi_2 \nabla^2 \psi_1 + (\nabla \psi_2 \cdot \nabla \psi_1) \} dV$, interchanging ψ_1 and ψ_2

$$\text{i.e. } \oint_S (\psi_1 \nabla \psi_2 - \psi_2 \nabla \psi_1) \cdot d\underline{S} = \iiint_V (\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1) dV \quad \left(\begin{array}{c} \text{Green's} \\ \text{Theorem} \end{array} \right)$$

This is a general relationship between scalar functions ψ_1 and ψ_2 .

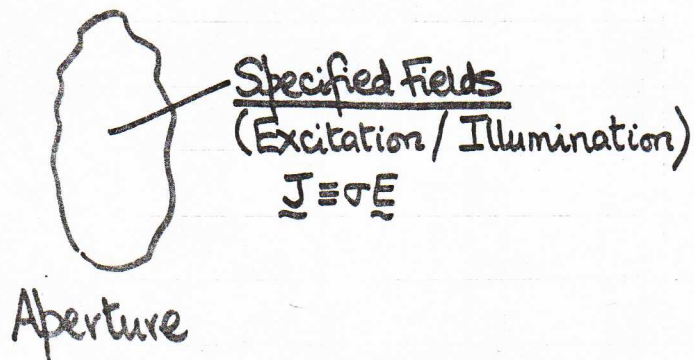
For solutions to the scalar Helmholtz Equation, $\nabla^2 \psi = -k^2 \psi$, so the R.H.S. = 0

Now put $\psi_1 = \psi$
and $\psi_2 = \frac{e^{ikr}}{r}$, which we know is a solution to the Helmholtz Eqn.

$$\text{Then } \oint_S \left(\psi \nabla \left(\frac{e^{ikr}}{r} \right) - \frac{e^{ikr}}{r} \nabla \psi \right) \cdot d\underline{S} = 0 \quad \begin{array}{l} \text{over any closed surface } S \\ \text{within which } (\nabla^2 + k^2) \psi = 0 \end{array}$$

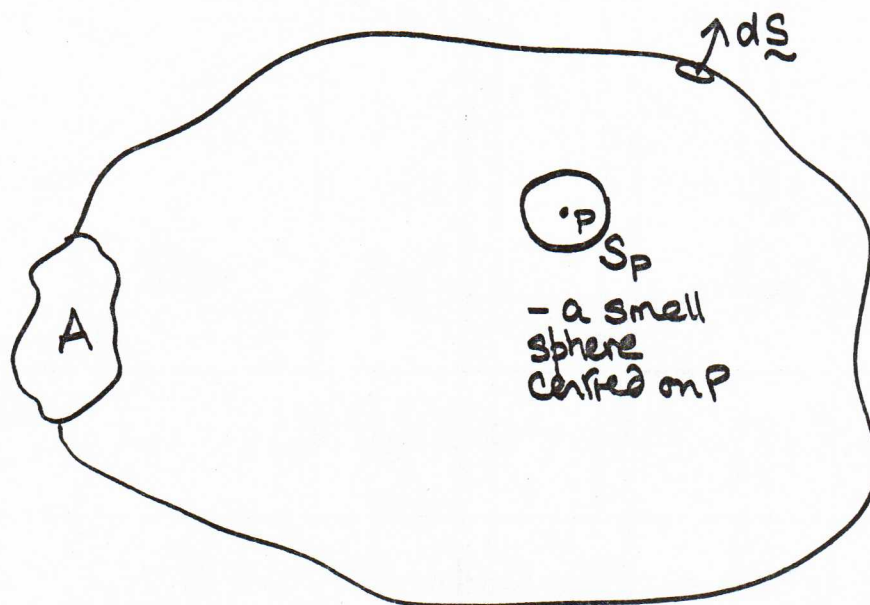
Now we are normally interested in the following problem :-

• Field Point P



Q. What is ψ_P ?

To tackle this, construct a surface Σ in two parts:



$$dS_p = r^2 d\Omega$$

radially out from S_p
towards P

S - any closed surface including the aperture A

$$\Sigma = S + S_p$$

Take S_p as a small sphere, radius r , centred on P
origin of co-ordinates at P ($r=0$ at P , $=r$ on S_p)

Then no sources within Σ and $\oint_{\Sigma} \left(\psi \nabla \left(\frac{e^{ikr}}{r} \right) - \frac{e^{ikr}}{r} \nabla \psi \right) \cdot d\mathbf{S} = 0$

is the same thing as

$$\oint_S \left(\right) = - \oint_{S_p} \left(\right)$$

Now consider $\oint_{S_p}$, is

$$\oint_{S_p} \left(\psi_p \left(\frac{ike^{ikr}}{r} - \frac{1}{r^2} e^{ikr} \right) \left(-\hat{r} \right) - \frac{e^{ikr}}{r} \nabla \psi_p \right) \cdot \hat{r} r^2 d\Omega$$

NB - gradient $\parallel -\hat{r}$

$$\oint_{S_p} \left(\psi_p \left(ikpe^{ikr} - e^{ikr} \right) - pe^{ikr} \left(\nabla \psi_p \cdot \hat{r} \right) \right) d\Omega$$

As $r \rightarrow 0$, $e^{ikr} \rightarrow 1$

$$\oint_{S_p} \rightarrow +4\pi \psi_p \left(\oint_{4\pi} \psi_p d\Omega \right)$$

Thus
$$\oint_S \left(\psi \nabla \left(\frac{e^{ikr}}{r} \right) - \frac{e^{ikr}}{r} \nabla \psi \right) \cdot d\vec{S} = -4\pi \psi_P$$
 when P is taken as origin of coordinates

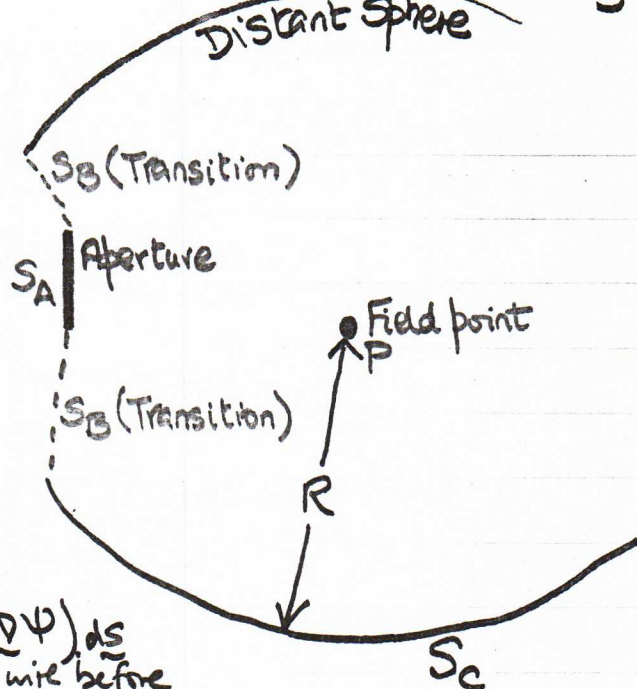
This is known as the Kirchhoff-Helmholtz Integral Theorem

The K-H theorem tells us how to evaluate ψ_P by integration involving ψ and $\nabla \psi$ over a surface that surrounds P .

The basis of scalar diffraction theory rests with the choice of the surface S that contains the aperture A

— by suitable choice we eliminate most of the contribution to the integral

Take a side view of the general situation:



Given the field point P, construct the surface of integration as follows:

S_A - the aperture area itself

S_C - a sphere of large radius R centred on P

S_B - a transition surface between A and C.

Now our $\oint_S = \left(\iint_{S_A} + \iint_{S_B} + \iint_{S_C} \right) \left(\Psi \nabla \frac{e^{ikr}}{r} - \frac{e^{ikr}}{r} \nabla \Psi \right) dS$
 by analogy with before

Consider $\iint_{S_C} = \frac{e^{ikR}}{R} \iint_{S_C} \left(\Psi_R + ikR\Psi_R - R \nabla \Psi_R \right) d\Omega$, where $R \rightarrow \text{large}, \Psi_R \rightarrow 0$

so $\iint_{S_C} \rightarrow e^{ikR} \iint_{S_C} \left(ik\Psi_R - \frac{\partial \Psi_R}{\partial R} \right) R d\Omega$

$\iint_{S_C} \rightarrow 0$ if $\left(ik\Psi_R - \frac{\partial \Psi_R}{\partial R} \right) R \rightarrow 0$ for large R . ~~But even for~~ $\propto \frac{e^{ikr}}{r}$ This holds for a spherical wave and for all practical cases, so we can safely put the integral over $S_C \rightarrow 0$ in practice. Other approach is to say not strict monochromatic, take $R > c(t-t^*)$, where t^* is when $\Psi(t)$ "turned on".

The $\iint$ over S_B cannot be dealt with so properly. In fact we assume that it $\rightarrow 0$ by asserting that we can choose the transition region so that Ψ and $\nabla \Psi$ are zero on S_B . $\boxed{\Psi_B = 0}$ may be true in diffraction problems where we can assert that B is a completely opaque screen, or in antenna problems where B can be chosen as the outside of a conductor excited on the inside to "feed" the aperture. What cannot be justified is ignoring the derivatives of Ψ at the boundaries $S_A \rightarrow S_B$. In fact $\Psi_A \rightarrow 0$ over a real S_B on scales of order λ , so neglecting $\iint_{S_B}$ is valid only if S_A extends over dimensions $\gg \lambda$. In this case the integral over S_A dominates even with large derivatives where $A \rightarrow B$, and

$\Psi_P \rightarrow \frac{1}{4\pi} \iint_{S_A} \left(\Psi_A \nabla \left(\frac{e^{ikr}}{r} \right) - \frac{e^{ikr}}{r} \nabla \Psi_A \right) \cdot d\vec{S}'_A$

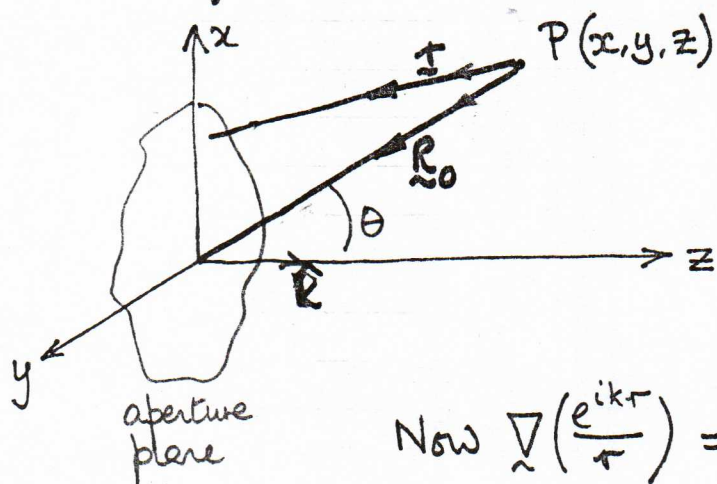
taking INWARD normal element to get rid of the $\ominus$ sign.

Note: Sign conventions in obs. factor may be screwed up through here.
 cthB 11/82

6/11

Now we will generally know $\Psi_A = A(x_A, y_A) e^{-ikF(x_A, y_A)}$ at $z=0$ aperture plane

$A \rightarrow$ amplitude distribution } across aperture
 $F \rightarrow$ phase distribution



$$\underline{r} = r \hat{r}$$

$R_0 \cdot R_0 \approx$ [N.B. not same $\hat{r}$ around P before]

$\left. \begin{matrix} A \\ F \end{matrix} \right\}$ specified by excitation

$$\text{Now } \underline{\nabla} \left(\frac{e^{ikr}}{r} \right) = \frac{e^{ikr}}{r} \left(ik - \frac{1}{r} \right) \hat{r}$$

$$\text{and } \underline{\nabla} \Psi_A = (\underline{\nabla} A) e^{-ikF} - ik A e^{-ikF} \underline{\nabla} F$$

$$= \Psi_A \left(\frac{1}{A} \underline{\nabla} A - ik \underline{\nabla} F \right)$$

$$\text{So that } \Psi_P = \frac{1}{4\pi} \iint_A \Psi_A \frac{e^{ikr}}{r} \left\{ \left(ik - \frac{1}{r} \right) \hat{r} + ik \underline{\nabla} F - \frac{1}{A} \underline{\nabla} A \right\} \cdot \underbrace{\hat{k}}_{d\hat{S}_A} dx_A dy_A$$

This is what we generally have to handle.

In many cases we will have plane wave illumination, or slightly modified plane wave

For these, $A \sim \text{constant}$ $\frac{1}{A} \underline{\nabla} A \rightarrow 0$ (negligible)

$F = +z$ for propagation towards field pt. ($e^{-i\omega t} e^{ikz}$)

~~$$\underline{\nabla} F = \hat{k}$$~~

$$ik \underline{\nabla} F = +ik \cdot \hat{k}$$

Sign?

~~Note the orientation of the co-ordinates based on the field point~~

Diffraction regimes may be divided into three "zones" of approximations to $\iint_A$:

1) Near Zone No further simplification — generally very messy!

2) Transition (Fresnel) Zone

① Neglect $\frac{1}{r}$ term

② Neglect variation of $\hat{r} \cdot \hat{k}$, put $= \hat{r} \cdot \hat{k} = \cos \theta$

③ Set $\frac{1}{r} = \frac{1}{r_0}$ everywhere

④ Neglect $\frac{1}{A} \nabla A$ cf. $ik \nabla F$ [if source of excitation is far from aperture in AA $\rightarrow$ plane wave]

Now expand r in terms of field and aperture coordinates:

$$\begin{aligned} r^2 &= (x-x_A)^2 + (y-y_A)^2 + z^2 \\ &= z^2 \left(1 + \left(\frac{x-x_A}{z} \right)^2 + \left(\frac{y-y_A}{z} \right)^2 \right) \end{aligned}$$

Close to axis (small-angle diffraction)

$$\begin{aligned} r &\rightarrow z + \frac{(x-x_A)^2}{2z} + \frac{(y-y_A)^2}{2z} \\ &= z + \Delta_A \quad [\text{defining notation } \Delta_A] \end{aligned}$$

Then $\Psi_P \Rightarrow \frac{ike^{ikz}}{4\pi R} \iint \Psi_A e^{ik\Delta_A} (\cos\theta + \nabla F \cdot \hat{\mathbf{r}}) dx_A dy_A$ signs?

The bracketed term is known as the **OBLIQUITY FACTOR**. It says that the contributions of different parts of the aperture to Ψ_P are weighted not merely by the variation of the amplitude factor $A(x_A, y_A)$ over the aperture and by the variation of the phase factor $k\Delta_A$, but also by a directionality depending on θ and on the form of the phase of the excitation.

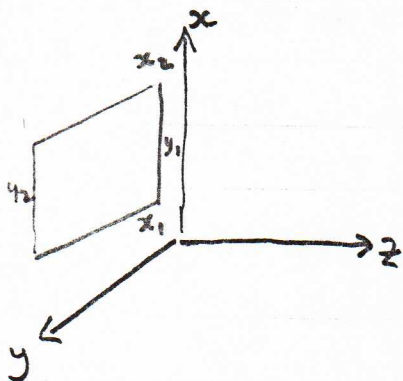
In many practical cases the aperture is effectively illuminated/excited by a plane wave, for which $F \rightarrow +z$, $\nabla F \rightarrow +\hat{\mathbf{z}}$, and the obliquity factor $\rightarrow (\cos\theta + 1)$. Note that this suppresses back propagation ($\theta = 180^\circ$), and maximises forward propagation, apart from any effects of the Δ_A phase factors. (Hence "mathematical equivalent of Huygens's Principle")

14) Fresnel Diffraction

In the Fresnel approximation we assume that $e^{ik\Delta_A}$ varies much faster with x_A and y_A than does the obliquity factor, and retain terms as far as second order in x_A, y_A in computing Δ_A .

ie. $\Psi_P \Rightarrow \frac{-ike^{ikz} (\cos\theta + \nabla F \cdot \hat{\mathbf{z}})}{4\pi R} \iint \Psi_A e^{ik\{(x-x_A)^2 + (y-y_A)^2\}/2z} dx_A dy_A$
 $\uparrow$ note this contains $A(x_A, y_A)$ in general.

Consider a simple case - plane illumination of a rectangular aperture



$$\Psi_P \rightarrow C \int_{x_1}^{x_2} \int_{y_1}^{y_2} e^{ik\{(x-x_A)^2 + (y-y_A)^2\}/2z} dx_A dy_A$$

where C contains: the obliquity factor

the wave amplitude $A = \text{constant}$

the propagation factors $\frac{e^{ikz}}{R}$

which we take to be constant in this approx.

Fresnel Integrals and Cornu Spiral

$$C(w) = \int_0^w \cos\left(\frac{\pi a^2}{2}\right) da$$

$$(a = (x-x_A)\sqrt{\frac{k}{\pi z}} \text{ or } (y-y_A)\sqrt{\frac{k}{\pi z}})$$

$$S(w) = \int_0^w \sin\left(\frac{\pi a^2}{2}\right) da$$

$$C(0) = S(0) = 0$$

$$C(-w) = -C(w) \text{ and } S(-w) = -S(w) \quad \text{ANTISYMMETRIC}$$

$$C(\infty) = S(\infty) = \frac{1}{2}$$

Take $C(w)$ and $S(w)$ as rectangular coordinates of a point $Q(w)$
Locus of $Q(w) \rightarrow$ CORNU SPIRAL

Length of line from any point $Q(w_1)$ to $Q(w_2)$ on the spiral
is $L^2 = [C(w_2) - C(w_1)]^2 + [S(w_2) - S(w_1)]^2$

Hence the one-dimensional cut through the diffraction pattern at given, e.g. y , consists of an intensity proportional to the square of the length of the L^2 corresponding to the given w_1, w_2 .

e.g. Diffraction by straight-edge $w_1 = -\infty$

$w_2 = 0$ at the axis of edge

w_2 -ve $\rightarrow$ in shadow

w_2 +ve $\rightarrow$ in "light"

The Cornu Spiral

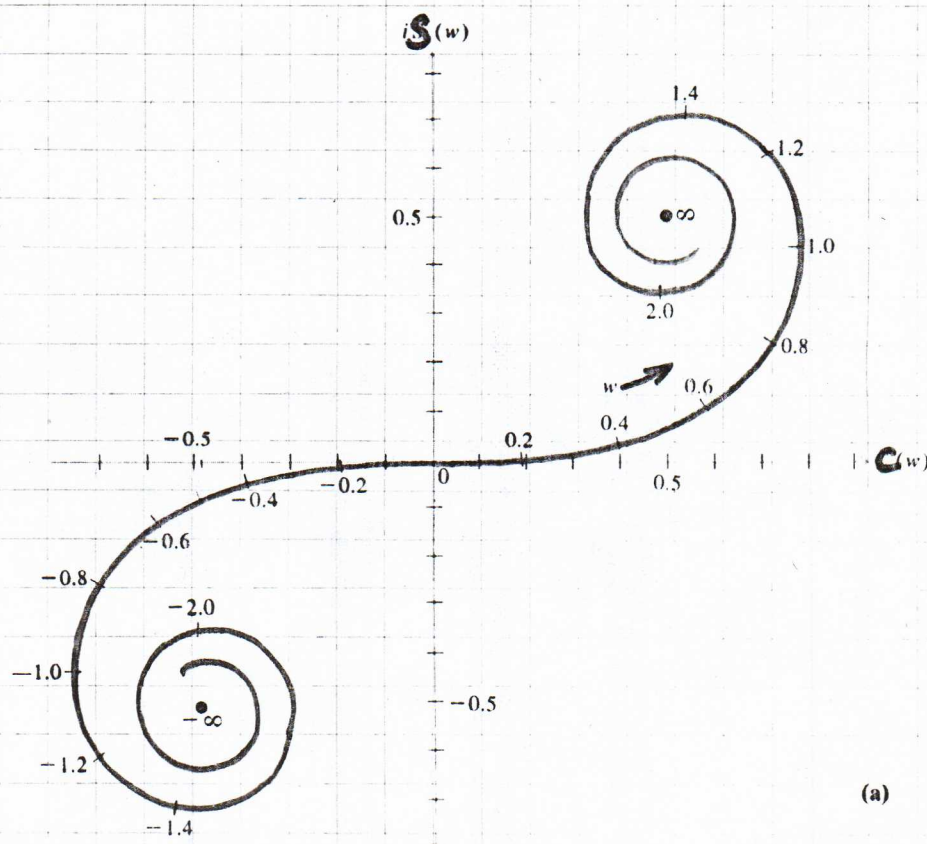


Figure 4.24. (a) The Cornu spiral. The scale of w is marked on the curve.

$$\begin{aligned}
 U_p &= \frac{U_0}{1+i} \left[C(v) + iS(v) \right]_{-x}^{v_2} \\
 &= \frac{U_0}{1+i} \left[C(v_2) + iS(v_2) + \frac{1}{2} + \frac{1}{2}i \right]
 \end{aligned}
 \tag{4.49}$$

which is a function of only the one variable v_2 . This variable specifies the position of the diffracting edge. If the receiving point P is exactly at the geometrical shadow edge, then $v_2 = 0$. We have then $U_p = U_0/(1+i)(\frac{1}{2} + \frac{1}{2}i) = \frac{1}{2}U_0$. Hence the amplitude at the shadow edge is one-half, and the intensity is one-fourth the unobstructed value. A plot of $I_p = |U_p|^2$ as given by Equation (4.49) is shown in Figure

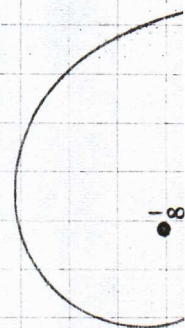


Figure 4.24. (b)

4.25. Here I_p is a fixed position diffracting edge. From the graph monotonically hand, in the i diminishing an The highest in point $v_2 \approx 1.2$ wave. This is

4.6 Application

Let us return to consider the general an arbitrary sl retardation, w

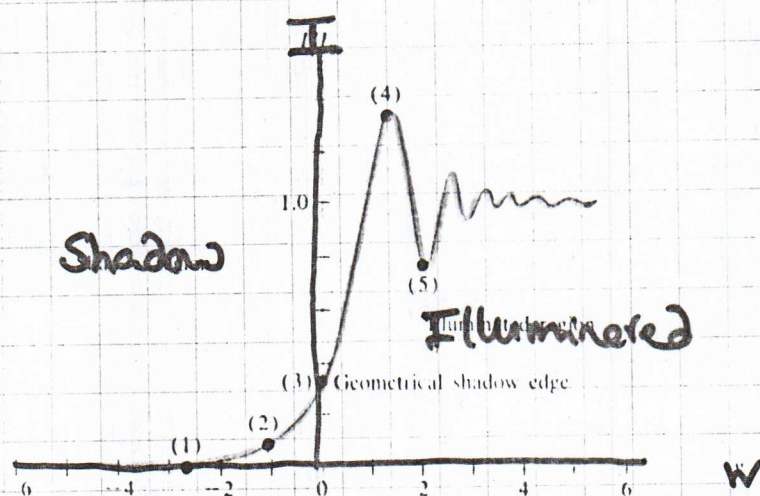
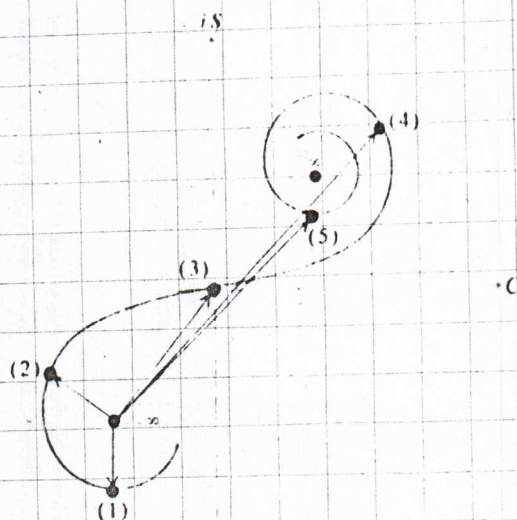


Figure 4.25. Fresnel diffraction by a straightedge.

We choose coordinates as indicated in Figure 4.26. The diffracting aperture lies in the xy plane, and the diffraction pattern appears in the XY plane, which is the focal plane of the focusing lens. According to elementary geometrical optics, all rays leaving the diffracting aperture in a given direction, specified by direction cosines α , β , and γ are

brought
where
assum
and β
No
 $Q(x, y)$
Figure
of the

It fol
giving
multip

This
we int
tion is
wave
Equat

The integrand can be evaluated in terms of the integral

$$\int_0^w e^{i\pi a^2/2} da = C(w) + iS(w) \quad (a \equiv (x-x_A)\sqrt{\frac{k}{\pi z}} + (y-y_A)\sqrt{\frac{k}{\pi z}})$$

$$\text{where } C(w) = \int_0^w \cos\left(\frac{\pi a^2}{2}\right) da \quad \text{and} \quad S(w) = \int_0^w \sin\left(\frac{\pi a^2}{2}\right) da$$

are known as the Fresnel Integrals.

A plot of $C(w)$ versus $iS(w)$ is known as the Cornu Spiral; this is a useful graphical visualisation of the basic integral. If the limit points w_1 and w_2 are marked on the spiral, the straight-line segment joining these points has a length proportional to the magnitude of the integral.

- e.g. (slides) diffraction by straight-edge $w_1 = -\infty$, $w_2 = w$.
 expand description diffraction at various distances from rectangular aperture.
 be explicit here

The Fresnel diffraction "zone" is characterised by the onset of "diffusion" of the fields outside the boundaries of "ray optics" propagation through the aperture. In the distant limit, the "form" of the aperture is lost in the diffracted radiation, and we make an (ill-defined) transition to the "far" zone, characterised by the "Fraunhofer" approximation.

15) Fraunhofer Diffraction

At sufficiently great distances from the aperture, we may approximate

$$\Delta_A \sim \frac{x^2 - 2xx_A}{2z} + \frac{y^2 - 2yy_A}{2z}$$

The terms we are neglecting are $x_A^2/2z$, $y_A^2/2z$. For this neglect to be reasonable we need $\frac{kx_A^2}{2z} \ll \pi$, i.e. $z \gg \frac{x_A^2}{\lambda}$, $\frac{y_A^2}{\lambda}$

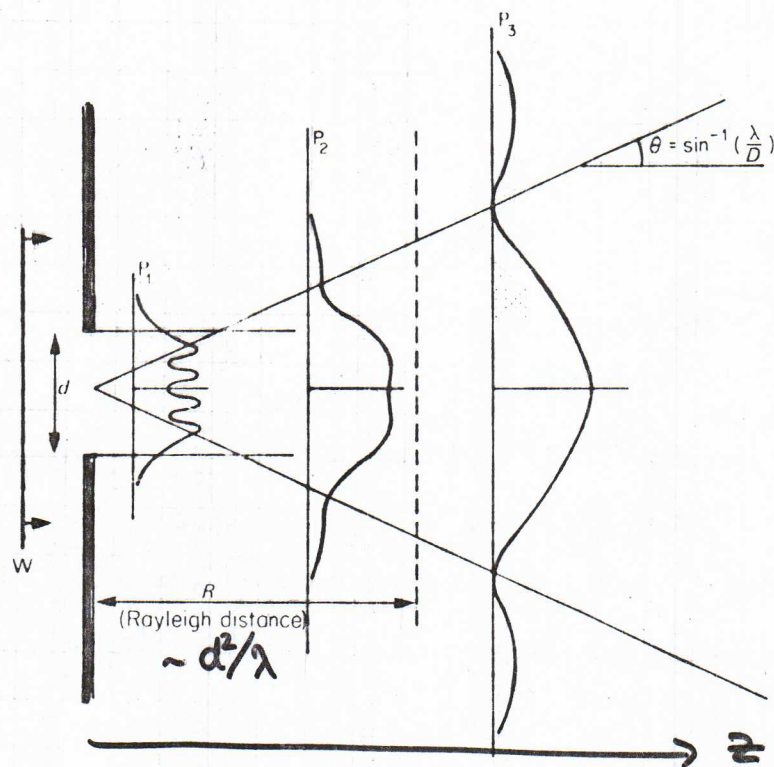


Fig. 16.1. Transition from Fresnel to Fraunhofer diffraction. A portion of a wave W passes through a slit, width d . Intensity distributions across the wave are shown for planes P_1 (close to the slit), P_2 (just inside the Rayleigh distance), and P_3 (beyond the Rayleigh distance).

(i) Different parts of the aperture may be at considerably different distances from P ; this variation may have a significant effect on wave amplitude as well as phase.

(ii) The line to P from different parts of the aperture may make considerably different angles with the normal to the surface of the aperture; this also may have a significant effect on amplitude.

These factors can be incorporated into a general diffraction integral through the Kirchhoff theory (Section 16.6). For the simplest case of a distant point source of illumination, r_0 from the aperture, the wave amplitude at the aperture can be written as $A_0 \exp(ikr_0)/r_0$. The diffracted wave amplitude $A(P)$ at P is given by the integral

$$A(P) = -\frac{ik}{4\pi} A_0 \frac{\exp(ikr_0)}{r_0} \int_{\text{aperture}} \frac{\exp(ikr)}{r} (1 + \cos \chi) da \quad (16.2)$$

Inside the in
each compo
to $1/r$, the d
inclination fa
Outside the
phase of $A(P)$
The evalu
a straight ed

16.2 SHAD STRA

One of the
is that there
ference fring
geometrical c
of the undis
shadow. Ou
undisturbed
proaches a u

Most prob
simplified m
It is natural

Fig. 16.2

For many purposes we can also confine attention to field points whose distance from the $\hat{z}$ -axis is small compared with z , i.e. to a restricted range of angles near the z -axis

$$\begin{aligned} \text{Then: } \frac{x}{z} &\rightarrow \frac{x}{R} = \sin\theta \cos\phi & \frac{x^2}{2z} &\rightarrow 0 \\ \frac{y}{z} &\rightarrow \frac{y}{R} = \sin\theta \sin\phi & \frac{y^2}{2z} &\rightarrow 0 \\ \text{and } \cos\theta &\sim 1 \end{aligned}$$

$$\text{and so: } \psi_p \rightarrow \frac{ik e^{ikR}}{4\pi R} (1 + \nabla \cdot \hat{z}) \iint_A \psi_A e^{-ik \sin\theta (x_A \cos\phi + y_A \sin\phi)} dx_A dy_A$$

where we have again taken the obliquity factor out of the integral on the presumption that it will be slowly-varying by comparison with $e^{ik\Delta A}$.

$$\begin{aligned} \text{If we define } k_x &= k \sin\theta \cos\phi \\ \text{and } k_y &= k \sin\theta \sin\phi \end{aligned}$$

$$\text{then } \psi_p \propto \Psi(k_x, k_y) \text{ where } \Psi(k_x, k_y) = \iint_A \psi_A e^{-i(k_x x_A + k_y y_A)} dx_A dy_A$$

Note: 1) Ψ is playing the same rôle as the amplitude of the Schellkunoff Vector.

2) If we imagine an INFINITE aperture extending to $\pm\infty$ outside the real aperture, and $\psi_A = 0$ outside the real aperture, then $\Psi(k_x, k_y)$ is the two-dimensional FOURIER TRANSFORM of the extended aperture illumination ψ_A . $\Psi(k_x, k_y) \longleftrightarrow \psi_A(x_A, y_A)$

i.e. the intensity distribution near the z -axis is proportional to the square of the angular Fourier spectrum of the spatial illumination/excitation ψ_A .

The simplicity of this result suggests a more intuitive approach:—

Angular and Spatial Spectra of Radiation

—see, e.g. Ratchliffe, *Rep. Prog. Phys.*, 19, 188 (1965)

Consider our aperture-illumination function as used in the theory:

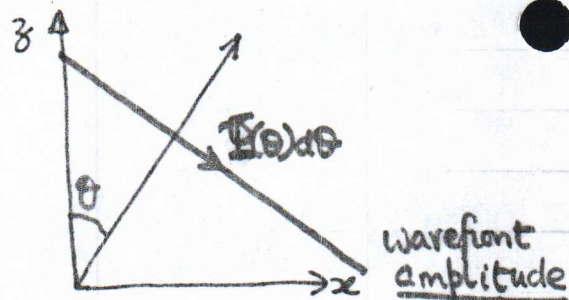
$$\psi_A = A(x, y) e^{-ikF(x, y)}$$

The fact that its angular power pattern in the far (Fraunhofer) field is related to the Fourier transform of this function can be thought of physically as follows:

Consider a 2-d problem, no variation of ψ_A with y , $\psi_A(x) = A(x) e^{-ikF(x)}$

Let a system of plane waves, all of frequency ω (= ω_0 to that of ψ_A) travel in various directions at angles θ to the z -axis:

Assume that these are transverse (vector) waves and that the complex amplitude of the wave in $(\theta, \theta + d\theta)$ is $\Phi(\theta) d\theta$



Then the wave travelling at θ is

$$\Phi(\theta) \exp i[k(x \sin \theta + z \cos \theta) - \omega t]$$

Now add all of these, taking their x -cpts to get a scalar function $\psi_x = \psi$

$$\rightarrow \psi(x, z, t) = \int_{-\pi/2}^{+\pi/2} \Phi(\theta) \cos \theta e^{i[kx \sin \theta + kz \cos \theta - \omega t]} d\theta$$

↑ taking the x -cpt

Now put $\sin \theta = S$, $\cos \theta = C$ rewrite $\Phi(\theta)$ as $\Phi(\sin \theta) = \Phi(S)$

$$\psi(x, z) = \int_{-1}^{+1} \Phi(S) e^{ik(Sx + Cz)} dS$$

At the aperture $z=0$,
$$\psi(x) = \int_{-1}^1 \Psi(s) e^{i2\pi s x / \lambda} ds$$

Now put $x = x/\lambda$ and consider formally $-\infty < s < \infty$

$$\psi(x) = \int_{-\infty}^{+\infty} \Psi(s) e^{i2\pi s x} ds$$

This is a Fourier Transform relationship $\psi(x) \leftrightarrow \Psi(s)$

$$\Psi(s) = \int_{-\infty}^{+\infty} \psi(x) e^{-i2\pi s x} dx$$

(Note the normalisation: $2\pi s$ is playing the rôle of ω before, i.e. $s \Leftrightarrow \nu$).

i.e. a spatial disturbance $\psi(x)$ is equivalent to a superposition of plane waves travelling in all directions, $\Psi(s)$ where $\psi \leftrightarrow \Psi$, at a given wavelength λ / frequency ω .

This (angular $\leftrightarrow$ spatial) correspondence is the basis of our radiation theory for the Fraunhofer regime. Given $\psi(x)$ ($= \psi(x/\lambda)$) in the aperture, the limiting case of the Fraunhofer pattern is that where at an angle θ we detect ONLY the θ -component of the angular spectrum. Note the rôle of a lens in $\rightarrow$ Fraunhofer diff. at finite $z \rightarrow \frac{L \sin \theta}{\lambda} \Leftrightarrow$

How about the meaning of the formal extension $|s| > 1$?

$\rightarrow$ this means $\cos \theta = \sqrt{1 - s^2} = i\gamma$

i.e. their z -variation is exponentially attenuated.

Consider a small bundle of waves in the angular spectrum at $|s| > 1$

$$\rightarrow \Psi(s) ds e^{ik(sx + cz)} \Rightarrow \Psi(s) ds e^{-k\gamma z} e^{iksx}$$

These correspond to the terms of SPATIAL FREQUENCY (λ_s) in the x -direction in the representation of $\Psi(x)$.

The math is therefore equivalent to saying that the effect of SPATIAL FREQUENCIES $< \lambda$ in $\Psi(x)$ is EXPONENTIALLY DAMPED in the z -direction, being effectively lost altogether from the plane-wave spectrum after a few wavelengths in z ($\exp -k\gamma z$).

Thus the radiation field cannot carry information on detail in $\Psi(x)$ smaller than a wavelength more than a few wavelengths away from the z coordinate at which the detail was imposed.

This is effectively the wavelength limit of practical microscopy.

In doing the integral $\int_{-\infty}^{+\infty} dS$ we are including the exponentially-damped information about structure in $\Psi(x)$ on a scale corresponding to spatial frequencies $<$ the wavelength of the radiation. This information does not propagate into the Fraunhofer regime, so $\int_{-\infty}^{+\infty} \equiv \int_{-1}^1$ for all practical purposes.

Two-dimensional analogue:

$$\Psi(x, y) = \iint_{-\infty}^{+\infty} \Phi(S_1, S_2) \exp 2\pi i(S_1 x + S_2 y) dS_1 dS_2 \quad \text{at } z=0$$

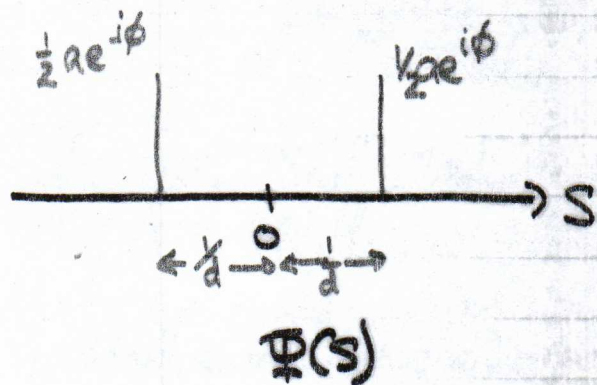
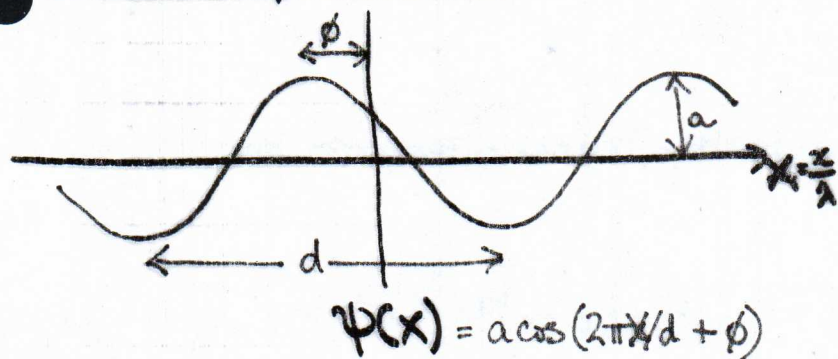
→ uniform plane waves travelling in directions specified by S_1, S_2

→ evanescent if $S_1^2 + S_2^2 > 1$.

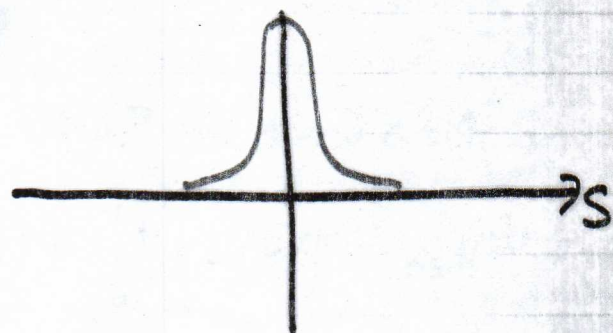
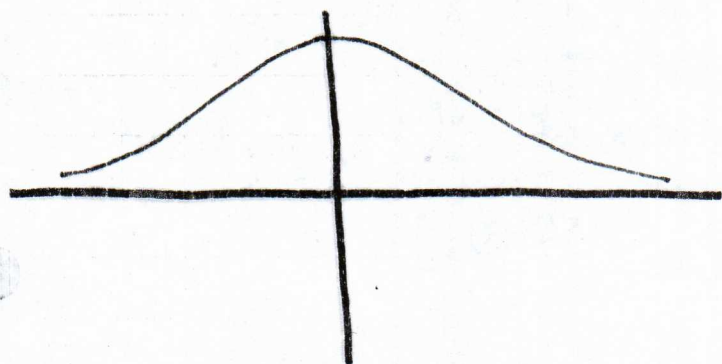
Circular symmetry in $\Psi(x, y) \rightarrow \Psi(r) = 2\pi \int_0^\infty \Phi(S) J_0(2\pi r S) S dS$

Hankel transform $\rightarrow \Phi(S) = 2\pi \int_0^\infty \Psi(r) J_0(2\pi r S) r dr$

Some useful representations :-



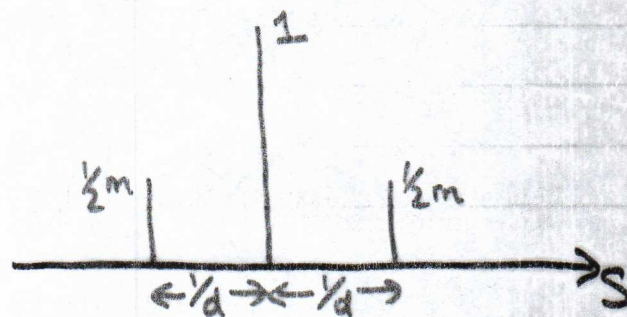
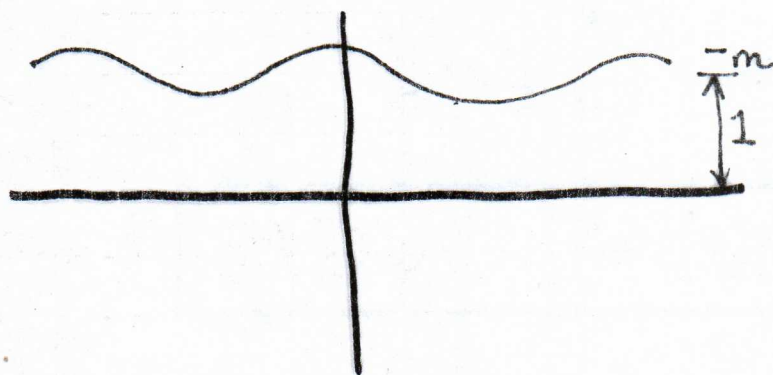
Invert these $\psi(x)$ = two narrow slits (Young's Slits) → cosinusoidal $\Phi(s)$



Gaussian $\psi(x) = \exp(-\pi x^2/d^2)$

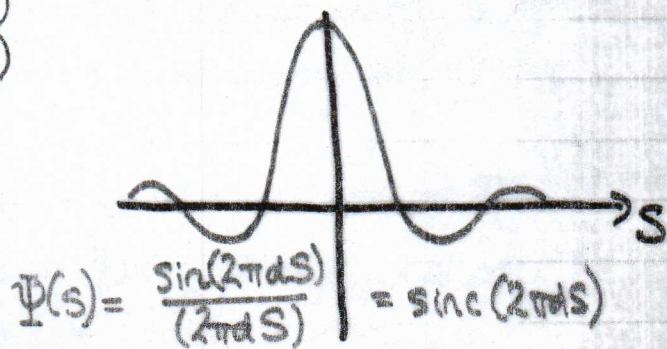
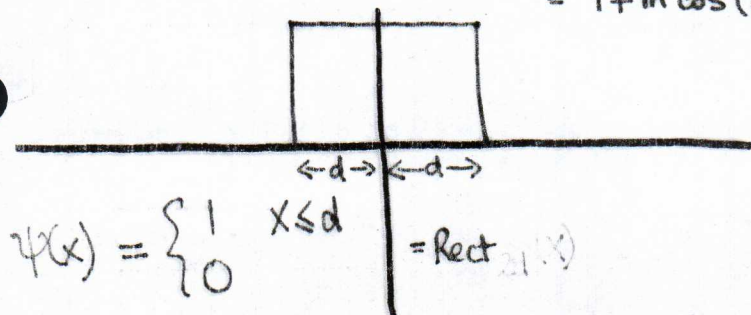
Note again the reciprocal scales.

$\Phi(s) = \exp(-\pi s^2 d^2)$



N.B. the Fourier transform
SUM RULE → this from example #1.

Amplitude Modulation $\psi(x) = 1 + m \cos(2\pi x/d)$
 $= 1 + m \cos(kx/d)$



Before proceeding further, let us develop a useful theorem for the Fourier Transforms — the CONVOLUTION THEOREM

Let $\psi_1(x) \leftrightarrow \Psi_1(s)$
 $\psi_2(x) \leftrightarrow \Psi_2(s)$ be two Fourier-Transform pairs

Consider a new function $\psi(x) = \psi_1(x) \cap \psi_2(x)$, where $\cap$ represents convolution of ψ_1 with ψ_2 .

$$\text{i.e. } \psi(x) = \int_{-\infty}^{+\infty} \psi_1(y) \psi_2(x-y) dy$$

Now write the Fourier representation of $\psi_2(x-y) = \int_{-\infty}^{+\infty} \Psi_2(s) e^{2\pi i s(x-y)} ds$

$$\text{then } \psi(x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi_1(y) [\Psi_2(s) e^{2\pi i s(x-y)}] ds dy$$

As s and y are independent, we may permute the order of integration:

$$\psi(x) = \int_{-\infty}^{+\infty} \Psi_2(s) e^{2\pi i s x} \left[\int_{-\infty}^{+\infty} \psi_1(y) e^{-2\pi i s y} dy \right] ds$$

$$\text{i.e. } \psi(x) = \int_{-\infty}^{+\infty} \Psi_2(s) e^{2\pi i s x} [\Psi_1(s)] ds$$

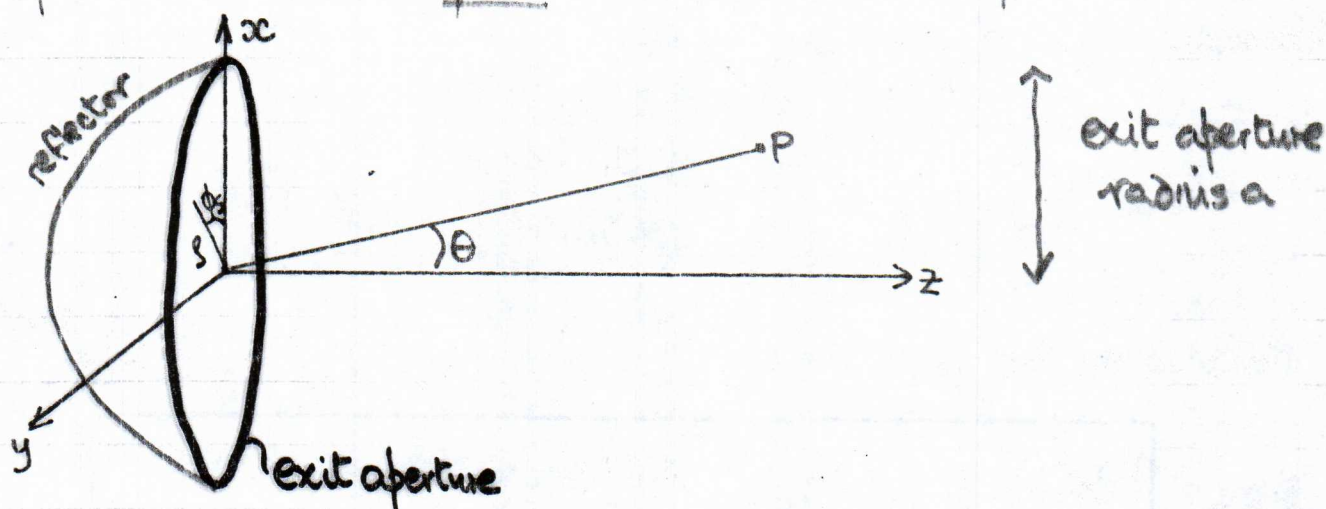
$$\text{i.e. } \boxed{\psi(x) = \psi_1(x) \cap \psi_2(x) \leftrightarrow \Psi_1(s) \Psi_2(s)}$$

i.e. the Fourier Transform of the convolution of two functions is the product of their individual Fourier Transforms.

This theorem will be important in analysing Fraunhofer diffraction patterns of arrays of apertures.

16) Fraunhofer Pattern of a Uniformly Illuminated Circular Aperture

e.g. a circular-aperture antenna, uniformly illuminated by a small "feed" which radiates a spherical wave over the angle subtended by the reflector. Parabolic geometry then converts this spherical wave to a plane wave at the exit aperture.



Field point coordinates r, θ, ϕ . By symmetry, the radiated Ψ cannot depend on ϕ , so compute at $\phi=0$ (P in the x, z plane).

Source point coordinates (ρ, ϕ_s)

$$x_A = \rho \cos \phi_s \quad k_x = k \sin \theta \cos \phi_s = k \sin \theta$$

$$y_A = \rho \sin \phi_s \quad k_y = k \sin \theta \sin \phi_s = 0$$

Uniform illumination $\rightarrow \Psi_A = 1$

$$\Psi(k_x, k_y) = \iint_A e^{-i(k \sin \theta \cdot \rho \cos \phi_s)} dS_A$$

$$\Psi(k_x, k_y) = \int_0^{2\pi} \int_0^a e^{-ik \rho \sin \theta \cos \phi_s} \rho d\rho d\phi_s$$

To evaluate this, we use the integral representation of the Bessel Function

$$J_n(z) = \frac{i^{-n}}{2\pi} \int_0^{2\pi} e^{iz \cos \alpha} e^{in\alpha} d\alpha$$

$$\therefore \Psi(k_x, k_y) \text{ here} = 2\pi \int_0^a J_0(k \rho \sin \theta) \rho d\rho$$

To proceed, use the recurrence relation for Bessel Functions:

$$\frac{d}{dz} \{ z^{n+1} J_{n+1}(z) \} = z^{n+1} J_n(z)$$

Put $n=0$, $\int_0^z \{ J_0(y) \} dy = z J_1(z)$

i.e. $\int_0^{k a \sin \theta} \frac{k p \sin \theta J_0(k p \sin \theta) d(k p \sin \theta)}{k^2 \sin^2 \theta} = \frac{k a \sin \theta J_1(k a \sin \theta)}{k^2 \sin^2 \theta}$

and $\Psi(k_x, k_y) = \frac{2\pi a J_1(k a \sin \theta)}{k \sin \theta}$
 $= 2\pi a^2 \left\{ \frac{J_1(k a \sin \theta)}{k a \sin \theta} \right\}$

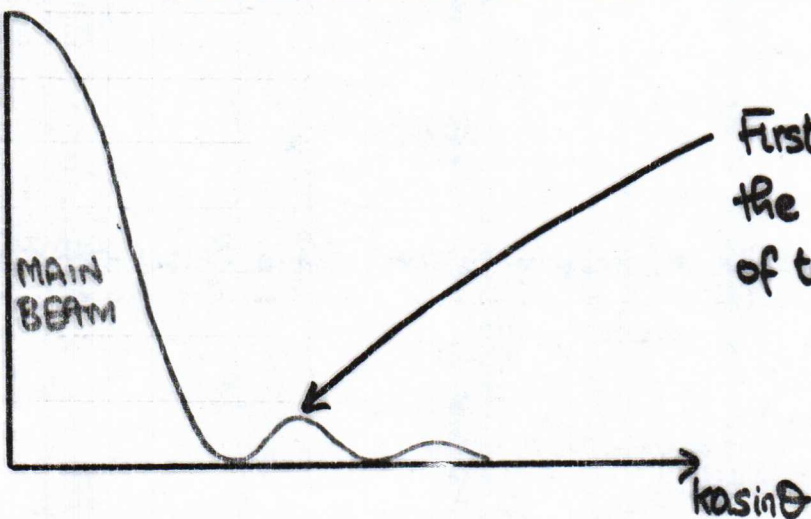
Hence $\frac{d\langle P \rangle}{d\Omega} \propto a^4 \left\{ \frac{J_1(k a \sin \theta)}{k a \sin \theta} \right\}^2$

HPBW = $1.02 (\lambda/2a)$

First zero is at $\theta = \sin^{-1} (1.22 \lambda/2a)$

Directivity ($a \gg \lambda$) = $\frac{4\pi^2 a^2}{\lambda^2} = \frac{4\pi A}{\lambda^2}$

$\frac{d\langle P \rangle}{d\Omega}$



First sidelobe (ring lobe around the main beam) contains ~ 7% of total power radiated

Effect of tapering the illumination of circular aperture

For some antenna applications, the effects of the secondary maxima in the radiation pattern (discrete sidelobes) may be undesirable. If a smoother variation in the angular coordinate θ is needed, we obviously need to construct a radiation pattern convolved with a function of finite width in θ . This corresponds to multiplying the aperture illumination Ψ_A by a function of finite width in a , whose Fourier transform is the desired smoothing function in $\sin\theta$. This adjustment is known as tapering the illumination, and can be carried out by appropriately redesigning the feed element.

It is usually desirable to minimise the power which misses the reflector, so the tapering function $T(\rho) \rightarrow 0$ for $\rho \rightarrow a$.

A common choice is $T(\rho) = (1 - \rho^2/a^2)^p$

$$\Psi \rightarrow \Psi_0 T \quad \Psi \rightarrow \Psi_0 \cap t(s) \text{ where } t(s) \leftrightarrow T(\rho)$$

This case turns out (after some math) to be analytic:

$$\Psi \rightarrow \pi a^2 \cdot \frac{2^p p! J_{p+1}(ka \sin\theta)}{(ka \sin\theta)^{p+1}}$$

p	RELATIVE GAIN	HPBW	θ_0^{first}	Power at peak of 1st sidelobe
0	1.00	$1.02(\lambda/2a)$	$\arcsin\left(\frac{1.22\lambda}{2a}\right)$	-17.6 dB
1	0.75	$1.27()$	$\arcsin\left(\frac{1.63\lambda}{2a}\right)$	-24.6 dB
2	0.56	$1.47()$	$\arcsin\left(\frac{2.03\lambda}{2a}\right)$	-30.6 dB

17) Arrays of identical elements (see, e.g. Marion, Ch. 8; Papas, Ch. 3)

In dealing with diffraction by gratings or with radiation from array antennas, we may need to examine the converse of the tapering calculation, in which the illumination is the convolution of two known functions and the radiation pattern is governed by the product of their Fourier transforms.

e.g. the performance of an array of circular apertures

$$\bigcirc \bigcirc \bigcirc \bigcirc \equiv \bigcirc \cap \text{---|---|---|}$$

$$\text{F.T.} = \text{F.T.} \bigcirc \times \text{F.T.} (\text{---|---|---|})$$

i.e. $\text{Array } dP/d\Omega = \text{Element } dP/d\Omega \times \text{Array Factor}$

The behaviour of arrays of identical elements reduces entirely to the separate computation of the element performance and of the array geometry factor.

In general we can write the array excitation as a comb with N "teeth"

$$a(x) = \sum_{n=0}^{N-1} a_n e^{in\gamma} \delta(x-n\Delta) \quad (\text{one-dimensional})$$

where $\begin{cases} a_n = \text{relative amplitude of } n\text{'th comb} \\ n\gamma = \text{relative phase of excitation of } n\text{'th comb} \quad (\text{progressive phase shift } \gamma) \\ \Delta = \text{physical comb separation} \end{cases}$

$$\text{Array Factor} = |A(s)|^2 \quad A(s) = \int a_n e^{in\gamma} \delta(x-n\Delta) e^{ikx \sin\theta \cos\phi} dx$$

$$\text{i.e. } A(s) = \sum_{n=0}^{N-1} a_n e^{in\gamma} e^{ikn\Delta \sin\theta \cos\phi}$$

$$= \sum_{n=0}^{N-1} a_n y^n \quad \text{where } y = e^i x, \quad \alpha = (k\Delta \sin\theta \cos\phi + \gamma)$$

Thus the array factors $\sim |A(s)|^2$ can be derived from a POLYNOMIAL characterising the array excitation. Array theory was equivalenced to polynomial theory by Schelkunoff, Bell Syst. Tech. J, 22, 80 (1943)

General character.

Uniform array, $a_n = 1$, $\Sigma \rightarrow \sum_0^{N-1} y^n = \frac{y^N - 1}{y - 1} = e^{i(N-1)\frac{\alpha}{2}} \frac{\sin(N\alpha/2)}{\sin(\alpha/2)}$

Array factor $\Rightarrow \left| \frac{\sin(N\alpha/2)}{\sin(\alpha/2)} \right|^2$

Binomial array $a_n = \frac{(N-1)!}{(N-1-n)!n!}$ (tapered amplitude distribⁿ) ($0! = 1$ here)

$\Sigma \rightarrow (1+y)^{N-1}$

Array factor $\Rightarrow 2^{2N-2} \cos^{2N-2}\left(\frac{\alpha}{2}\right)$ - better sidelobes at the expense of less directivity.

$\gamma=0, k\Delta=\pi \Rightarrow 2^{N-2} \cos^{2N-2}\left(\frac{\pi}{2} \sin\theta \cos\phi\right)$ - NO SIDELOBES.

Chebyscheff array, based on Chebyscheff polynomial $C_n(x) = \cos(n \cos^{-1} x)$
optimises the sidelobe level for given directivity.

Main types of uniform array:

$\gamma=0, \Delta < \lambda \rightarrow$ single major lobe broadside to array line

$\gamma=0, \Delta > \lambda \rightarrow$ multilobe pattern like optical grating

$\gamma = \pm k\Delta \rightarrow$ radiation cast along line of sources
 $\rightarrow$ end-fire array

$k\Delta < \pi \rightarrow$ unilateral (single-lobe)

$k\Delta \geq \pi \rightarrow$ bilateral (two-lobe)

$\gamma = \beta t \rightarrow$ phase-scanning, shifts maximum in time without scanning physical array.

Directivity of arrays

e.g. uniform array of N half-wave dipoles

$$\frac{d\langle P \rangle}{d\Omega} = \frac{15}{\pi} \left| \frac{\cos(\frac{\pi}{2} \sin \theta)}{\sin \theta} \cdot \frac{\sin(N(k\Delta \sin \theta \cos \phi + \delta)/2)}{\sin((k\Delta \sin \theta \cos \phi + \delta)/2)} \right|^2$$

↓
slow θ -variation

↓
for large N , fast θ, ϕ variation

$$\left. \frac{d\langle P \rangle}{d\Omega} \right|_{\max} \text{ is at } \theta = \frac{\pi}{2}, \phi = \cos^{-1} \left(\frac{-\delta}{k\Delta} \right)$$

Broadside ($\delta=0$) directivity = $\frac{4\pi d\langle P \rangle/d\Omega}{\langle P \rangle} \xrightarrow{N \rightarrow \infty} \frac{4N\Delta}{\lambda}$ Virtually regardless of array element

first sidelobe power $\Rightarrow 0.045$ main lobe regardless of N for N large

main lobe width between zeros $\Rightarrow 2\lambda/N\Delta$

Uniform Rectangular (XY) array Directivity $\Rightarrow \frac{2\pi(N_x\Delta_x)(N_y\Delta_y)}{\lambda^2} \quad (N_{x,y} \rightarrow \infty)$

Directivity $\Rightarrow \frac{2\pi A}{\lambda^2}$

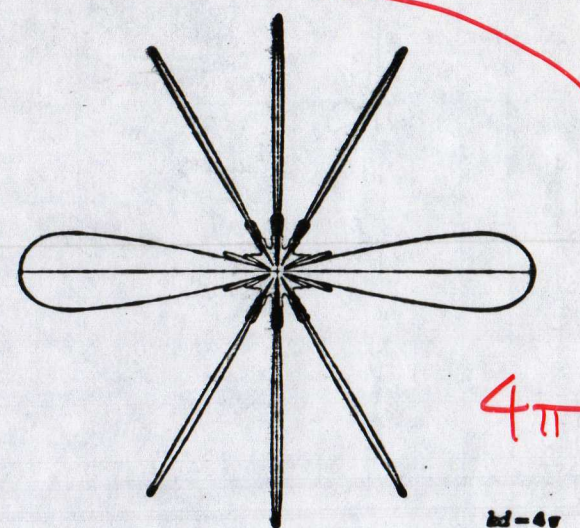
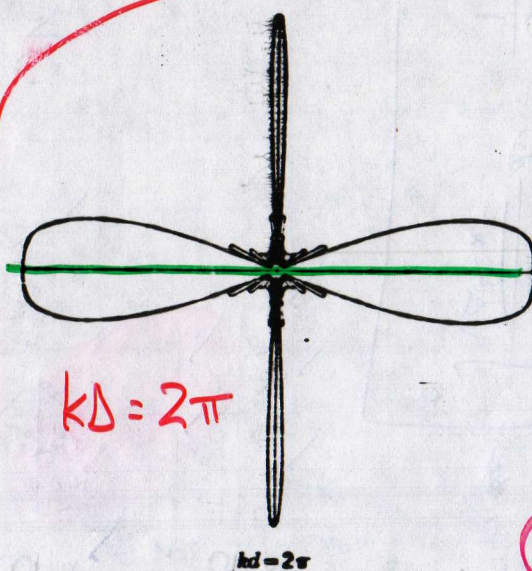
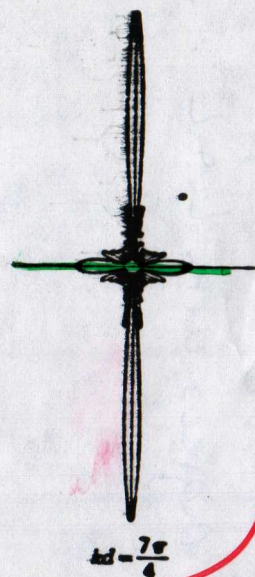
Non-uniform arrays, define "effective area" = $\frac{\lambda^2}{2\pi} \times \text{Directivity}$.

Explicitly, let an antenna array have $\Psi(x) = [\underbrace{\Psi_e(x)}_{\text{element}} \underbrace{A(x)}_{\text{uniform array factor}}] \cdot \underbrace{T(x)}_{\text{grading/tapering factor}}$

→ then, its far field (Fraunhofer) pattern derives from:-

$$\Psi(s) = [\Psi_e(s) \cdot \Psi_A(s)] \cdot \Psi_T(s)$$

e.g. chop the end off a waveguide  feeds. What happens?



GRATING
ARRAYS

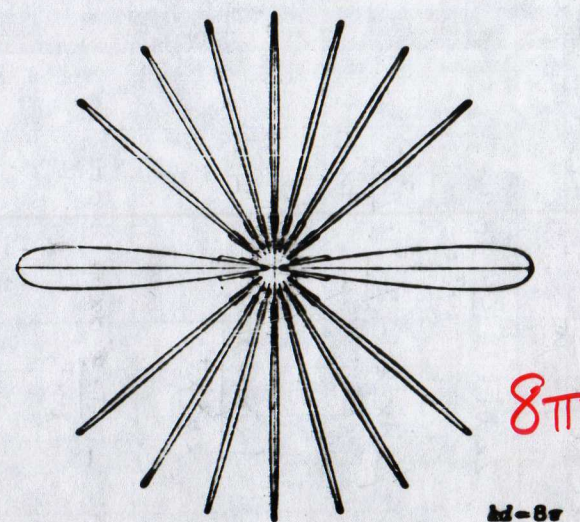
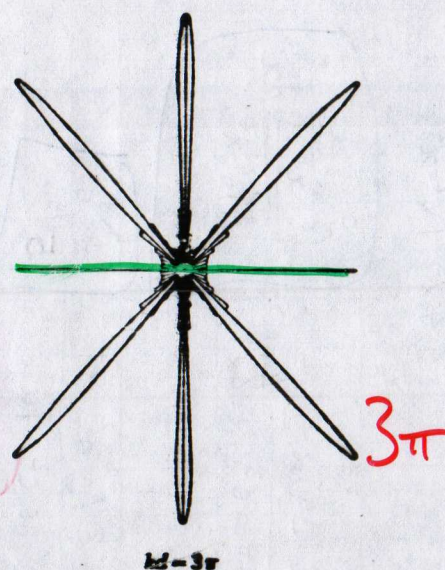
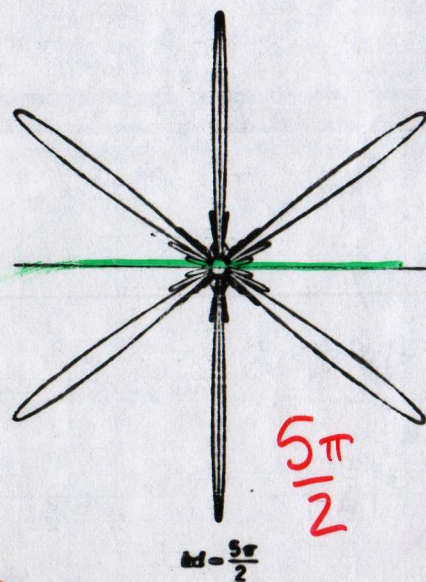


Fig. 3.8 Continued.

Fig. 3.8 Continued.

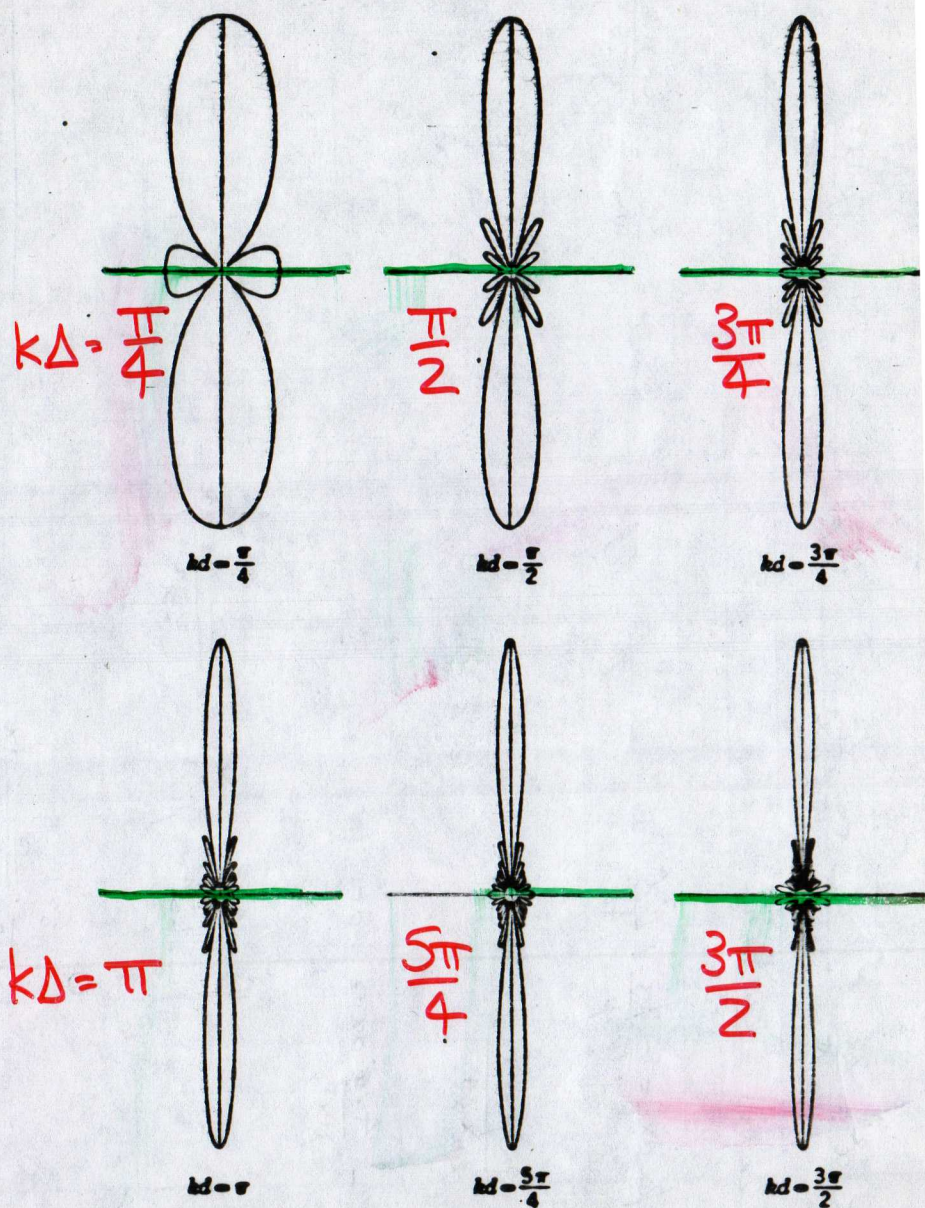
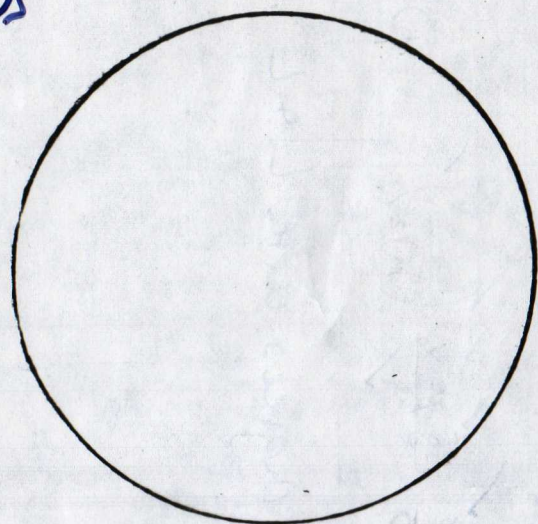
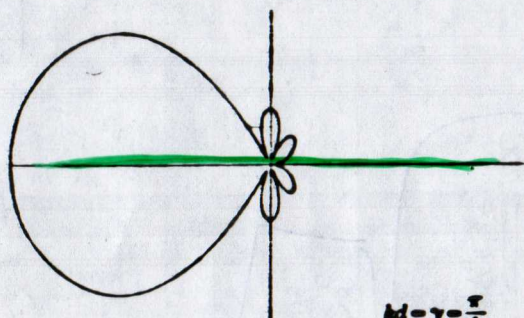


Fig. 3.8 Radiation characteristic $K(\psi)$ of a uniform linear array for various spacings. Calculated from Eq. (80) with $\gamma = 0$ and $n = 12$. Broadside array. Grating lobes.

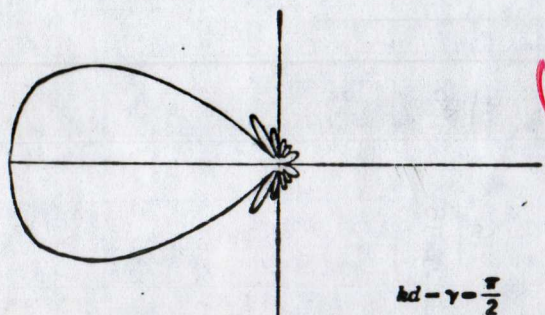
$$= \pm k\Delta$$



$$kd-\gamma=0$$



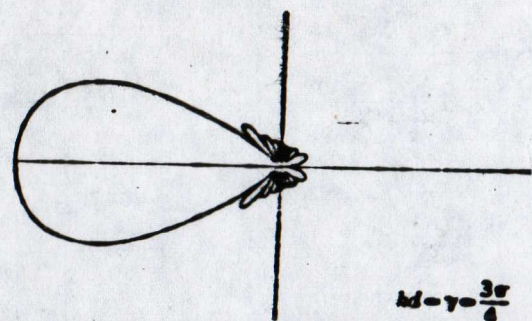
$$kd-\gamma=\frac{\pi}{4}$$



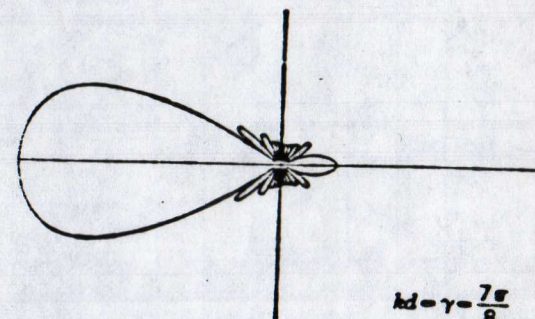
$$kd-\gamma=\frac{\pi}{2}$$

UNILATERAL
END-FIRE

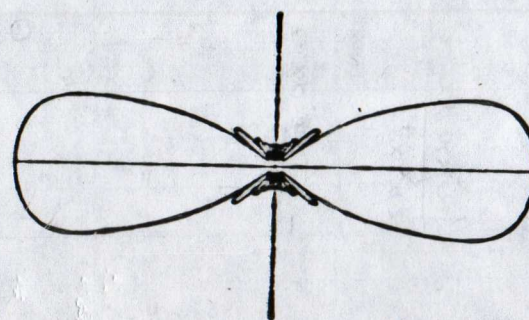
Fig. 3.9 Radiation characteristic $K(\psi)$ of a uniform linear array calculated from Eq. (80) with $n = 12$ for various values of $kd = \gamma$ Unilateral end-fire array. Bilateral end-fire array.



$$kd-\gamma=\frac{3\pi}{4}$$



$$kd-\gamma=\frac{7\pi}{8}$$

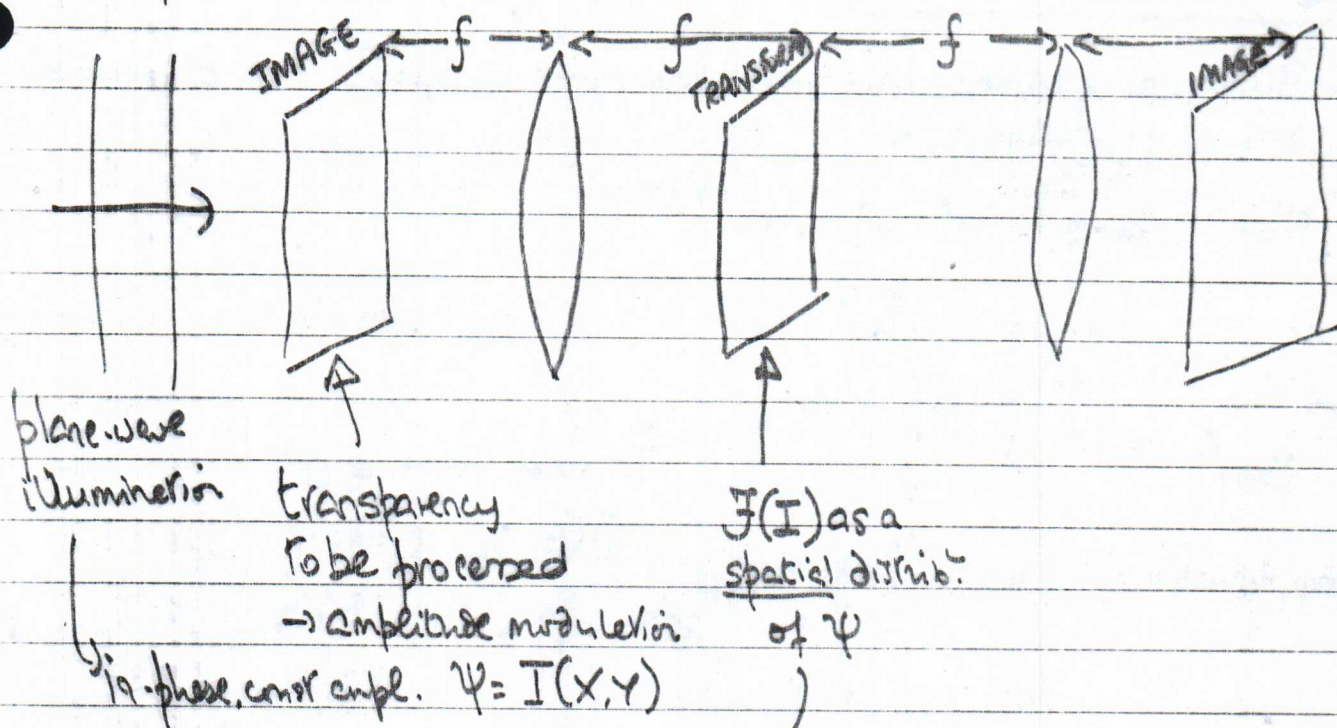


$$kd-\gamma=\pi$$

BILATERAL
END-FIRE
 $K\Delta = \gamma = \pi$

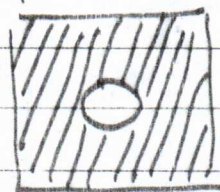
Fig. 3.9 Continued.

18) Spatial filtering in Image Processing



$F(I)$ can be filtered here to suppress unwanted angular frequencies (spatial frequencies in the original image plane). Spatial frequencies in the transform plane correspond to angular frequencies in the image plane — i.e. have a RECIPROCAL relationship to spatial frequencies in the image plane.

Mask to filter out high-freqs in image
i.e. to suppress short-range



inserted in Transform Plane

defect in the image, such as dot structure in a half-tone reproduction

~~Note that very high freqs. are also~~

Edge enhancement → needs mask, filter out the low-freqs in the image and leave the sudden contrasts.

Phase-Contrast Microscopy (Phase filtering)

Biological specimens have small variations in opacity but appreciable variations of refractive index

→ Phase screens $\Psi(x, y) = e^{iF(x, y)}$

Normal microscopy shows you $\Psi\Psi^* = 1$. Can't see the specimen unless it is STAINED, etc.



In focal plane we get

$$\Psi(s_1, s_2) \leftrightarrow \Psi(x, y)$$

$$\Psi(s_1, s_2) = \iint e^{iF(x, y)} e^{is_1x + is_2y} dx dy$$

If $F(x, y)$ is small,

$$\Psi \sim \mathcal{F}\{1 + iF(x, y)\}$$

$$\Psi = \delta(s_1, s_2) + if(s_1, s_2)$$

bright central spot

$f = \mathcal{F}(F)$ 90° out of phase



Now introduce a phase filter that introduces a phase shift of $\pi/2$ on the small-angle components (the central spot), but not on the large-angle components → filtered spectrum:—

$$\Psi' \rightarrow i\delta(s_1, s_2) + if(s_1, s_2)$$

Look at this with 2nd lens → output plane $\Psi(x, y) = \mathcal{F}(\Psi')$

$$\text{Now take output intensity } \Psi\Psi^* = |\mathcal{F}(\delta(s_1, s_2) + f(s_1, s_2))|^2$$

$$\sim 1 + 2F(x, y) + F^2(x, y) \leftarrow \text{small}$$

→ Amplitude variations in image depending on phase variations in object.

thicker parts of object appear brighter in output plane.

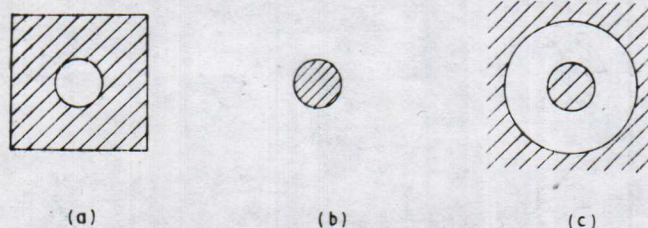


Fig. 6.14. (a) Low-pass filter, (b) high-pass filter, (c) band-pass filter.

1. Binary Filters. These are the simplest kind of filters; stops and slits are examples of this class of filters. The transmittance of such filters is either unity or zero. They can be further subdivided into the following categories:

a. Low-Pass Filters. These filters pass only the low frequencies, i.e., those frequency components lying near the axis (see Fig. 6.14a). These find application in filtering high-frequency noise. Thus for example, if one looks closely at printed photographs in a newspaper, one can see numerous dots to be printed on the image (see Fig. 6.15a). The dot structure in the photograph can be removed by placing a circular aperture centered around zero frequency in the Fourier transform plane as shown in Fig. 6.13. The circular aperture acts as a low-pass filter and the high spatial frequencies associated with the dot structure are filtered out. On the other hand, the spatial frequencies associated with the object are concentrated around

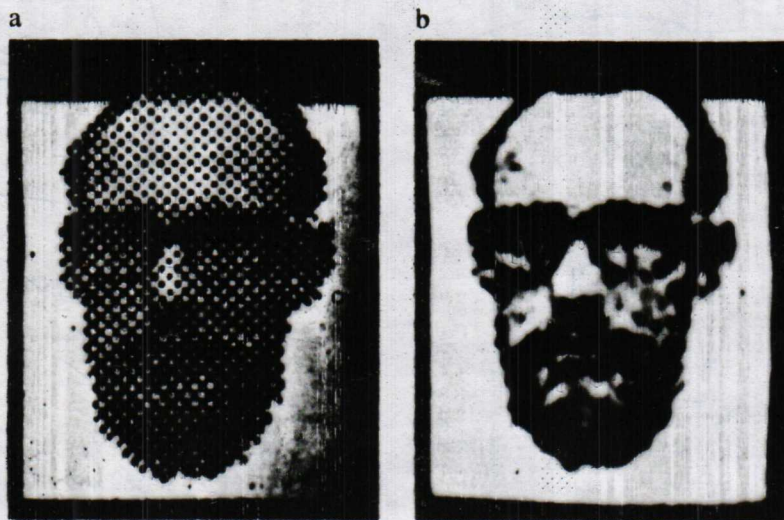


Fig. 6.15. Halftone dot structure removal using a low-pass filter (after Phillips, 1969).

Photograph courtesy of Dr. R. A. Phillips.

ORIGINAL
IMAGE WITH
HIGH SPATIAL FREQS.

FILTERED
IMAGE

the scene of the
when they do
prints are ma

6.5.4. Multi

Optical systems in the object plane, which forward many using optical By making u curvature along a per i.e., many on example, if o of functions, then using th correlations s tional to the plane P_1 (see



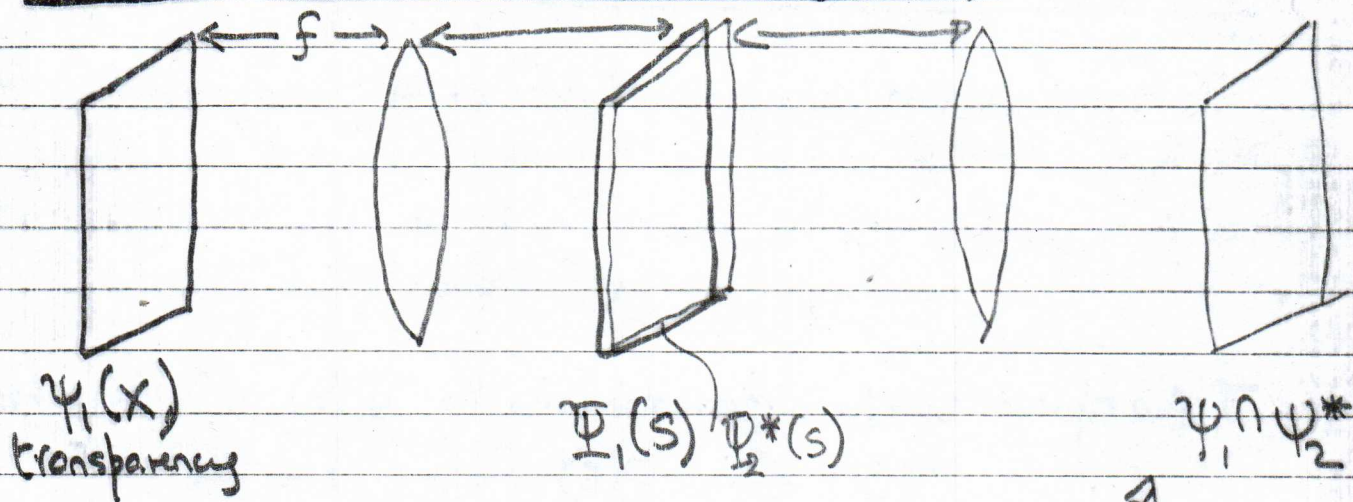
Fig. 6.20. Identification of fingerprints by cross-correlation technique (after Tsujiuchi *et al.*, 1971).

graphically prepared filter corresponding to the letter P is placed in plane P_2 (see Fig. 6.17), then in plane P_3 one obtains the intensity distribution shown in Fig. 6.19. The presence of a bright spot on the lower right corner (which represents a high correlation) indicates that the letter P appears in the corresponding position in plane P_1 . The convolution (which appears in the upper portion of the figure) and the cross-correlation with the letters Q and W produce a smear. The central portion corresponds to the first two terms on the right-hand side of Eq. (6.5-20). In a similar manner, the correlation technique can be applied to fingerprint identification. Figure 6.20 shows the output of the correlator when the fingerprint obtained from



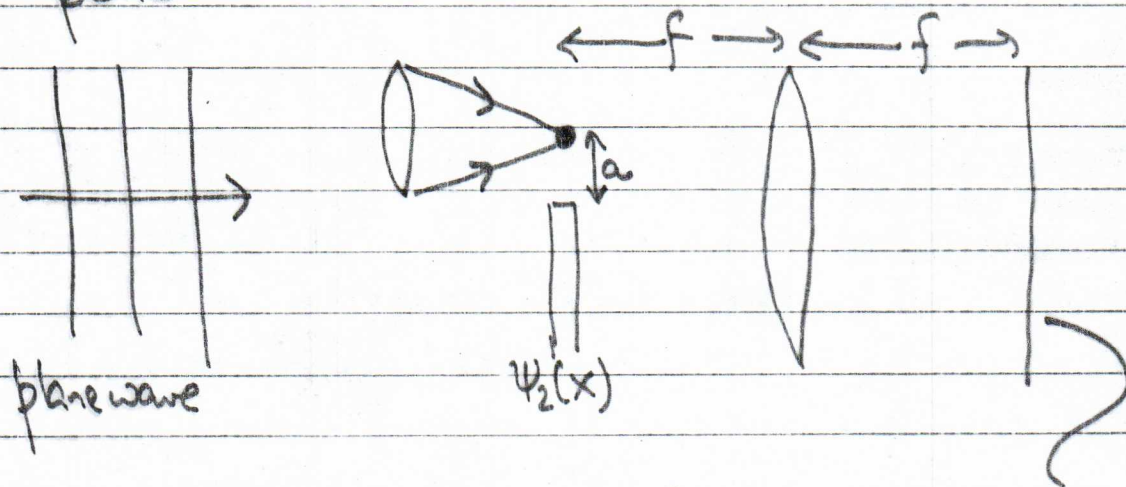
Cross-Correlation (Character Recognition)

77



Bright at displacements
where ψ_1 corresponds
to ψ_2

Q. how to produce the complex mask $\Psi_2^*(x)$ or the transform plane?



this has ψ_2 added to Ae^{iax}

i.e. amplitude in output plane = $\Psi_2(x) + Ae^{iax}$

intensity in output plane contains cross terms whose phase
contains phase of $\Psi_2(x)$ displaced by phase of the modulation.

→ offset ~~the~~ cross-correlation function of output plane.

19) The Wiener-Khintchine Theorem.

This is an important byproduct of the convolution theorem for Fourier Transforms. Suppose we have a wave field with an angular spectrum $\Psi(s)$ in its fields. The power spectrum

$$\langle P \rangle(s) \propto \Psi(s) \Psi^*(s)$$

If the spatial spectrum corresponding to $\Psi(s)$ is $\psi(x)$ then

$$\psi(x) \longleftrightarrow \Psi(s)$$

$$\psi(x) \wedge \psi^*(x) \longleftrightarrow \Psi(s) \Psi^*(s) \propto \langle P \rangle(s)$$

↑

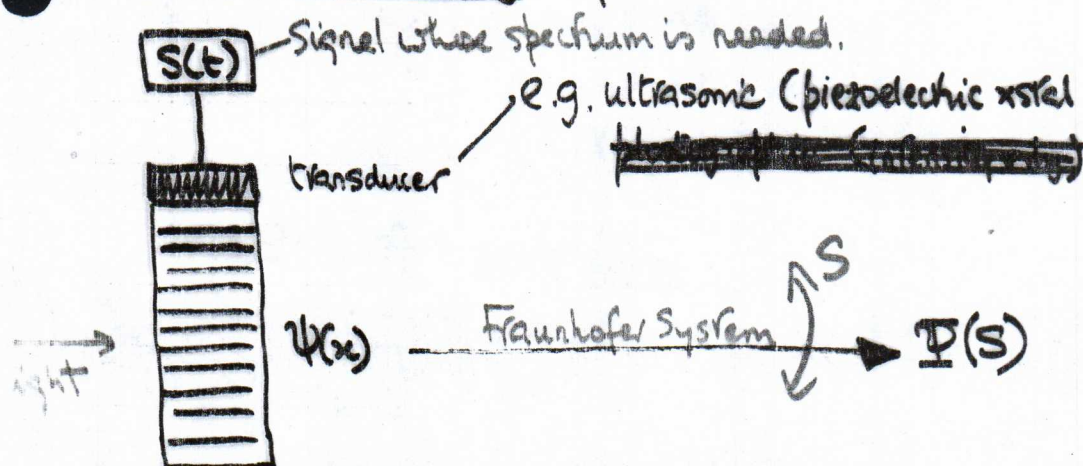
self-convolution of $\psi(x)$, or "AUTOCORRELATION" OF $\psi(x)$

The Wiener-Khintchine Theorem states that the power spectrum of a variable in one domain is ~~is~~ proportional to its autocorrelation function in the transform domain.

Consequences of the W-K Theorem

- 1) To achieve a given power pattern in an antenna design we must produce an aperture ~~illumination~~ illumination with a given spatial autocorrelation fn. Because different spatial distributions can have the same autocorrelation fn, antenna/^{power} pattern synthesis is not unique.
- 2) In X-ray diffraction we measure a three-dimensional/^{angular} power spectrum of scattered X-rays $\rightarrow$ is the three-dimensional spatial autocorrelation function of the distribution of X-ray scatterers (the electrons in the structure). Non-uniqueness of X-ray structures again results from non-uniqueness of autocorrelation fns.

Application in Spectroscopy (esp. microwave)



$$\text{If } \begin{matrix} \psi(\omega) & \longleftrightarrow & \Psi(s) \\ S(t) & \longleftrightarrow & s(\omega) \end{matrix} \quad \Psi(s) \propto S(\omega)$$

Need to make amplitude of ψ code amplitude of S

Easiest to produce small phase modulation (e.g. density contrast in pressure wave)

$$\psi = \psi_0 e^{i c S} \sim \psi_0 (1 + i c S) \text{ if } c S \ll 1 \text{ radian}$$

$$\sim \underbrace{\psi_0}_{\text{Steady signal}} + i \underbrace{\psi_0 c S}_{\text{modulation is } \propto \text{ for } S(t)}$$

Usually detect $\Psi^*(s) \rightarrow$ autocorrelation function of $\psi(x) \rightarrow S(t)$.
This is ok, gives the power spectrum of $S(t)$ (not the "field spectrum").

This is usually what you want, in practice. How much power at ω in $S(t)$?

Note the use of a lens can turn S spectrum into an spatial spectrum in image plane by turning plane waves to different points. Lens $\equiv$ going to $\infty \rightarrow$ lens is a Fourier Transformer over small range of ω (aperture-limited).

Imaging System Using Wiener-Khinchine Theorem



Incident Radiation
Power spectrum $P(S_1, S_2)$

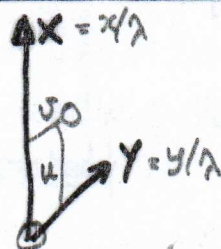


image plane E-fields $E(X, Y)$

The Wiener-Khinchine Thm $\rightarrow P(S_1, S_2) \leftrightarrow E(X, Y) \cap E^*(X, Y)$

i.e. if we sample $E(X, Y)$ in the image plane we can construct samples of its spatial autocorrelation function

$$p(u, v) = \int_{-\infty}^{+\infty} E(X, Y) E^*(X+u, Y+v) dX dY$$

from pairs of measurements with detectors separated by u in X , v in Y

Then Fourier-transform $P(S_1, S_2) \leftrightarrow p(u, v) \rightarrow$ image of $P(S_1, S_2)$

In practice we cannot sample $E(X, Y)$ at a point, but must sample over a finite aperture. I.e. we sample $E(X, Y) R(X, Y)$ where $R(X, Y)$ is a spatial response fn. So we measure $p(u, v) R^2(X, Y)$ whose Fourier transform is $P(S_1, S_2) \cap r(S_1, S_2)$ — the true radiation "image" multiplied by the power response in angle (power pattern diagram) of our detector $\rightarrow$ slowly-varying "attenuation" of edges of image.

Note that this requires measurement of ampl. and phase of $E(X, Y)$ — so need to use field-linear detectors such as heterodyne systems — applications are in radio, radar, infrared at present.

Side-looking radar. Don't measure $E(X)$, $E(X+u)$ simultaneously measure and record $E(t)$ as aircraft flies convert to $E(x)$, then analyse.

Aperture Synthesis in radio astronomy.

$u_{\max}, -v_{\max} \rightarrow$ RESOLUTION of image

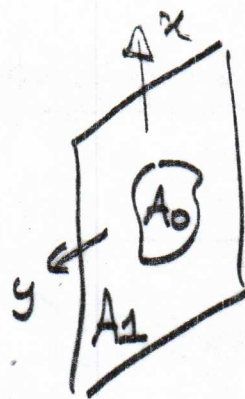
$R(X, Y) \rightarrow$ PRIMARY ATTENUATION of image.

Babinet's Principle

Suppose we have plane opaque screen in xy plane

- area A_1 covered by screen

A_0 covered by an aperture in screen



let there be sources in the $z < 0$ region of space

→ ψ_1 at points in the A_1 aperture

ψ_0 — — — — — A_0 aperture.

To calculate the radiation at field pt P passing through A_0 we would

compute ψ_P^0 from $\Phi_0(k_x, k_y)$ integrating over A_0 $\Phi_0 \xleftrightarrow{A_0} \psi_0$

Now suppose A_0 was opaque and A_1 was the aperture.

then ψ_P^1 from $\Phi_1(k_x, k_y) \xleftrightarrow{A_1} \psi_1$

If there were no screen ψ_P^t is from $\psi_t(k_x, k_y) \xleftrightarrow{\int_{A_0+A_1}} (\psi_0 + \psi_1)$

But $\int(\psi_0 + \psi_1) = \int(\psi_0) + \int(\psi_1)$ so

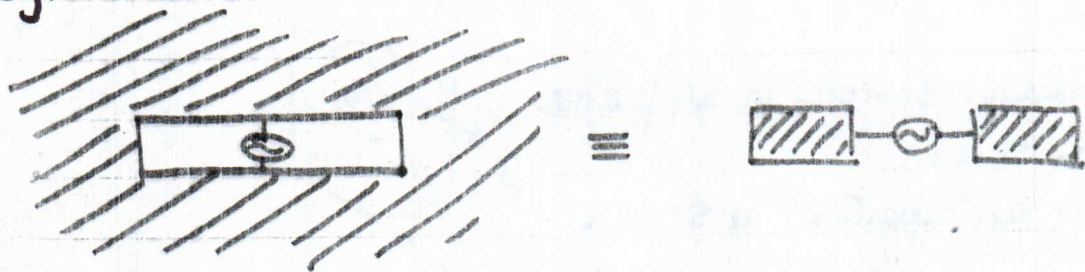
$$\psi_P^0 + \psi_P^1 \text{ must} = \psi_P^t$$

e.g. plane-wave illumination. $\psi_P^0(\theta \neq 0) \text{ must} = -\psi_P^1(\theta \neq 0)$ as $\psi_P^t(\theta \neq 0) = 0$

i.e. the power patterns of complementary apertures are identical, except at $\theta = 0$.

Ch. theory → $E \leftrightarrow H$ is the complementary patterns } Jackson
i.e. ψ^H is 90° different.

82
e.g. "Slot Radiators"



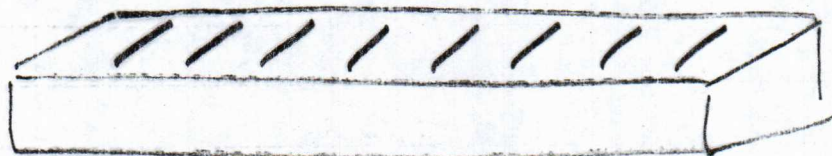
Slot in plane conducting sheet, energised by generator across the narrow dimension

=

Dipole made out of thin metal strip, energised by generator along the long dimension.

→ Waveguide array.

Slotted guide → convenient way of building high-frequency arrays:



~~Will treat other types of waveguide antenna later.~~

Simple open guide → know the field pattern in the waveguide mode
Fourier transform it
→ radiation pattern at large distances.

Vector Theory

Jackson
Stratton and Chu, Phys. Rev., 56, 99 (1939)

Solⁿ of the vector Helmholtz Eqns. with use of vector Green's Theorem

$$\rightarrow \underline{E}_P = -\frac{1}{4\pi} \iint_{S_A} \left\{ \frac{e^{ikr}}{r} \frac{\partial \underline{E}}{\partial n} - \underline{E} \frac{\partial}{\partial n} \left(\frac{e^{ikr}}{r} \right) \right\} dS + \frac{1}{4\pi} \oint_{\Gamma_A} \left[\frac{e^{ikr}}{r} (\underline{E} \times \underline{\tau}) - \nabla \left(\frac{e^{ikr}}{r} \right) (\underline{\tau} \cdot \underline{H}) \right] d\ell$$

(after much manipulation)

Where $\frac{\partial}{\partial n}$ is differentiation w.r.t. space $\perp$ to aperture A.

$\underline{\tau}$ is unit vector tangential to curve Γ_A bounding the aperture A.

Suppose for definiteness that $\underline{E}$ in the aperture is linearly polarised
so that $E_y = E_z = 0$.

The surface integral only $\rightarrow E_x$ at P. No $E_y(P)$ or $E_z(P)$ can arise

The $\underline{E} \times \underline{\tau}$ term is longitudinal, for small θ its contribution to E_x is $\sim \sin \theta$
Therefore $\underline{E} \times \underline{\tau}$ term $\rightarrow$ negligible contrib to $E_x(P)$ if θ is small.

The $\underline{\tau} \cdot \underline{H}$ terms from opposite sides of aperture would give contributions
to field in direction $\nabla(e^{ikr}/r)$. $\rightarrow$ i.e. longitudinal. It turns out
that the longitudinal components of $\underline{E}(P)$ arising from the boundary
integrals virtually cancel one another at small θ .

So at small θ , the vector theory gives results which $\rightarrow$ those of the scalar
theory.

Material Media - Adjustment to the field Equations

When we apply $\underline{E}$ and $\underline{B}$ to material media we may see responses due to the microstructure of the media which result from polarisation and magnetisation — either induced by the applied fields or derandomised by them.

Normally model only the electric and magnetic dipoles which appear in the microstructure, using macroscopic averages

$\underline{P}$ = electric dipole moment / unit volume

$\underline{M}$ = magnetic dipole moment / unit volume

$$\text{Then } \Delta \phi(\underline{r}) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{\rho_{\text{free}}(\underline{r}_s)}{|\underline{r} - \underline{r}_s|} + \frac{\underline{P}(\underline{r}_s) \cdot (\underline{r} - \underline{r}_s)}{|\underline{r} - \underline{r}_s|^3} \right\} \Delta V$$

gives the potential $\Delta\phi$ at a field point $\underline{r}$ outside a small volume ΔV around $\underline{r}_s$ that contains a free charge density $\rho_{\text{free}}(\underline{r}_s)$ and polarisation $\underline{P}(\underline{r}_s)$

$$\text{Hence } \phi(\underline{r}) = \frac{1}{4\pi\epsilon_0} \iiint \left\{ \frac{\rho_{\text{free}}(\underline{r}_s)}{|\underline{r} - \underline{r}_s|} + \underline{P}(\underline{r}_s) \cdot \nabla_s \left(\frac{1}{|\underline{r} - \underline{r}_s|} \right) \right\} dV$$

gives the potential outside a volume distribution of sources

$$\begin{aligned} \text{i.e. } \phi(\underline{r}) &= \frac{1}{4\pi\epsilon_0} \iiint \left\{ \frac{\rho_{\text{free}}(\underline{r}_s)}{|\underline{r} - \underline{r}_s|} + \nabla_s \cdot \left(\frac{\underline{P}(\underline{r}_s)}{|\underline{r} - \underline{r}_s|} \right) - \frac{1}{|\underline{r} - \underline{r}_s|} \nabla_s \cdot \underline{P}(\underline{r}_s) \right\} dV \\ &= \frac{1}{4\pi\epsilon_0} \left[\iiint_V \frac{\rho_{\text{free}} - \nabla_s \cdot \underline{P}(\underline{r}_s)}{|\underline{r} - \underline{r}_s|} dV + \iint_S \frac{\underline{P}(\underline{r}_s) \cdot d\underline{S}}{|\underline{r} - \underline{r}_s|} \right] \end{aligned}$$

The surface integral can be eliminated by taking V just larger than the medium's volume.

So $\phi(r) \rightarrow \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho'(r_s)}{|r-r_s|} dV$

where $\rho'(r_s) = \rho_{\text{free}}(r_s) - \nabla_s \cdot \underline{P}(r_s)$

The effects of volume polarisation can be included by replacing ρ_{free} by ρ' in the free-space relations.

Hence we expect the local relationship

$$\nabla \cdot \underline{E} = \rho_{\text{free}}/\epsilon_0 \rightarrow \frac{\rho_{\text{free}} - \nabla \cdot \underline{P}}{\epsilon_0}$$

i.e. $\nabla \cdot (\epsilon_0 \underline{E} + \underline{P}) = \rho_{\text{free}}$

Hence define $\underline{D} = \epsilon_0 \underline{E} + \underline{P}$ and say $\nabla \cdot \underline{D} = \rho_{\text{free}}$.

Similarly $\frac{4\pi}{\mu_0} \Delta \underline{A}(r) = \left[\frac{\underline{J}(r_s)}{|r-r_s|} + \frac{\underline{M}(r_s) \times (r-r_s)}{|r-r_s|^3} \right] \Delta V$

results in $\underline{J}' = \underline{J}_{\text{free}} + \nabla \times \underline{M}$

Hence define $\underline{H} = \frac{\underline{B}}{\mu_0} - \underline{M}$ and say $\nabla \times \underline{H} = \underline{J}_{\text{free}} + \dot{\underline{D}}$

The microstructure of the medium must define constitutive relations:

$$\left. \begin{aligned} \underline{B} &= \mu \underline{H} \\ \underline{D} &= \epsilon \underline{E} \\ \underline{J} &= \sigma \underline{E} \end{aligned} \right\} \text{these may be } \underline{\text{tensor}} \text{ relations.}$$

20) Propagation in Regions containing no Net Charge

$$\begin{aligned}\underline{\nabla} \cdot \underline{D} &= 0 & \underline{\nabla} \times \underline{E} &= -\dot{\underline{B}} \\ \underline{\nabla} \cdot \underline{B} &= 0 & \underline{\nabla} \times \underline{H} &= \sigma \underline{E} + \dot{\underline{D}} = \sigma \underline{E} + \epsilon \dot{\underline{E}}\end{aligned}$$

a) Isotropic Media μ, ϵ, σ constants and permute with $\underline{\nabla}$

$$\rightarrow \underline{\nabla} \times (\underline{\nabla} \times \underline{E}) = -\underline{\nabla} \times \dot{\underline{B}} = -\mu \underline{\nabla} \times \dot{\underline{H}} = -\mu (\sigma \dot{\underline{E}} + \epsilon \ddot{\underline{E}})$$

Take Fourier representations: $\underline{E}(t) = \int_{-\infty}^{+\infty} \underline{e}(\omega) e^{-i\omega t} d\omega$

$$\underline{B}(t) = \int_{-\infty}^{+\infty} \underline{b}(\omega) e^{-i\omega t} d\omega$$

$$\underline{\nabla} \times (\underline{\nabla} \times \underline{e}) = +i\omega\mu\sigma \underline{e} + \omega^2\mu\epsilon \underline{e}$$

$$\text{i.e. } \underline{\nabla} (\underline{\nabla} \cdot \underline{e}) - \nabla^2 \underline{e} = (i\omega\mu\sigma + \omega^2\mu\epsilon) \underline{e}$$

$$\text{i.e. } (\nabla^2 + k^2) \underline{e} = 0 \text{ where } \boxed{k^2 = \omega^2\mu\epsilon + i\omega\mu\sigma}$$

$$\text{Similarly } (\nabla^2 + k^2) \underline{b} = 0$$

i.e. we recover a vector Helmholtz Equation where the free-space relation $k^2 = \omega^2\mu_0\epsilon_0 = \omega^2/c^2$ is now modified.

Generally define $k^2 = (\omega^2/c^2) n^2$ where n = "refractive index" of the material medium.

$$\boxed{n^2 = \frac{\mu\epsilon + i\mu\sigma/\omega}{\mu_0\epsilon_0}} \text{ in a general medium (isotropic)}$$

$$\text{Dielectric } \sigma=0, \quad \boxed{n^2 = \frac{\mu\epsilon}{\mu_0\epsilon_0} = \mu_r\epsilon_r}$$

Good Conductor ($|J| \gg |D|$)

$$n^2 \Rightarrow \frac{i\mu\sigma}{\omega\mu_0\epsilon_0}$$

$$n \Rightarrow \frac{1+i}{\sqrt{2}} \sqrt{\frac{\mu\sigma}{\omega\mu_0\epsilon_0}} = c(1+i) \sqrt{\frac{\mu\sigma}{2\omega}}$$

Consider plane wave $\propto e^{i\omega(nz/c - t)}$

$$n = n_r + in_i \rightarrow e^{i\omega n_r z/c} e^{-\omega n_i z/c}$$

$n_r \rightarrow$ propagation

$n_i \rightarrow$ attenuation

$$\text{extinction coeff } \kappa = \frac{\omega n_i}{c} = k_i$$

Field decreases by $1/e$ in $\Delta z = \delta$ where $\frac{\omega n_i \delta}{c} = 1$

i.e. in the "Skin Depth"

$$\delta = \frac{c}{\omega n_i} = \sqrt{\frac{2}{\omega \mu \sigma}}$$

To determine the form of ϵ, μ, σ we need to examine microscopic average equations of motion of charges. Convenient to split this into examination of free electrons (plasma effects) and bound electrons (bound harmonic oscillators in classical theory), in presence of propagating radiation.

Propagation in Isotropic Plasma - Jackson, Ch. 7 Papas, Ch. 6
Marion, Ch. 9

- could describe propagation in ionised gas or in a metal.

- 1) assume only electrons displaced from equilibrium significantly, as $m_e \ll m_{\text{ions}}$.
- 2) neglect ionic polarisability (Simplification)
- 3) neglect magnetic properties of neutral atoms/ions ($\mu_r = 1$)
- 4) neglect forces on electrons due to B in wave ($F_m \sim 1/c F_e$ as before)
- 5) include radiative damping $\Gamma_r = \frac{e^2 \omega^2}{6\pi\epsilon_0 c^3 m}$
collisional damping $\Gamma_c = f_c$ - electron-ion collision freq.
($\dot{p} = -m_e f_c$)
 $\rightarrow \Gamma = \Gamma_r + \Gamma_c$

Then the free-electron equation of motion is

$$\ddot{\underline{r}} + m\Gamma\dot{\underline{r}} = \frac{q\underline{E}_{\text{wave}}}{m}$$

where $\underline{r}$ is the AVERAGE displacement of an electron from the position it would occupy in the absence of an incident wave.

The ω -frequency component of $\underline{r}(t) = \int_{-\infty}^{+\infty} \underline{r}(\omega)e^{-i\omega t} d\omega$ satisfies

$$\text{then } \underline{r}(t) = \frac{+q\underline{E}(\omega)e^{-i\omega t}}{m(-i\omega\Gamma - \omega^2)} + \text{transient}(\propto e^{-t/2})$$

$$\underline{r}(\omega) = \frac{-q\underline{E}}{m(\omega^2 + i\omega\Gamma)}$$

Now suppose that there are N free electrons per unit volume.
The ω -frequency component of the dipole moment per unit volume

$$\underline{P}(\omega) = Nq\underline{r}(\omega)$$

$$\text{i.e. } \underline{P}(\omega) = \frac{-Nq^2\underline{E}(\omega)}{m(\omega^2 + i\omega\Gamma)}$$

Given the definitions $\underline{D} = \epsilon_0\underline{E} + \underline{P} = \epsilon_r\epsilon_0\underline{E}$

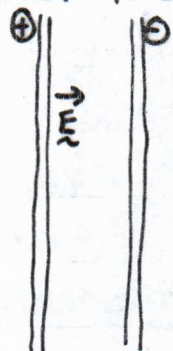
$$\text{it follows that } \underline{P} = (\epsilon_r - 1)\epsilon_0\underline{E}$$

$$\text{so } \underline{P}(\omega) = (\epsilon_r(\omega) - 1)\epsilon_0\underline{E}(\omega)$$

So in free-electron plasma

$$\epsilon_r(\omega) = 1 - \frac{Nq^2}{m\epsilon_0(\omega^2 + i\omega\Gamma)}$$

Now the quantity $Nq^2/m\epsilon_0 = \omega_p^2$, where ω_p is the natural frequency of undamped oscillations of the free-electron plasma, in the presence of the restoring force due to the ions. To see this consider displacing every free electron by the same x in an infinite slab of plasma. This creates a charge sandwich with surface charge density $\pm Nqx$ on each side, and a restoring E -field $E = -Nqx/\epsilon_0$. Hence on release the equation of motion is



$$\ddot{x} = -\frac{Nq^2x}{m\epsilon_0} = -\omega_p^2 x$$

$$\epsilon_r = 1 - \frac{\omega_p^2}{\omega^2 + i\omega\Gamma}$$

Scalar dielectric constant of free electron medium with damping.

Note i) Dispersion $\epsilon_r(\omega)$

ii) Real part $\epsilon_r(\text{real}) = 1 - \frac{\omega_p^2}{\omega^2 + \Gamma^2}$

→ simple refractive properties.

iii) Imaginary part $\epsilon_r(\text{im}) = \frac{+\omega_p^2\Gamma}{\omega(\omega^2 + \Gamma^2)}$

Going back to vector Helmholtz Eqn. $(\nabla^2 + \mu_0\epsilon_0\omega^2\mu_r\epsilon_r + i\omega\sigma\mu)\vec{E} = 0$

See that (for $\mu_r = 1$) $\mu_0\epsilon_0\omega^2\epsilon_r(\text{im}) \equiv +i\sigma\mu$

$$\epsilon_r(\text{im}) \equiv +\frac{\sigma}{\omega\epsilon_0}$$

i.e. plasma has equivalent CONDUCTIVITY

$$\Gamma = \frac{\epsilon_0\omega_p^2\Gamma}{\omega^2 + \Gamma^2}$$

i.e. damped free electron gas behaves like a lossy dielectric whose properties are strongly frequency-dependent.

● Regime $\omega \gg \omega_p, \Gamma$ ("HIGH" FREQUENCIES)

$$\epsilon_r \sim 1 - \frac{\omega_p^2}{\omega^2}$$

↓

$\Gamma \rightarrow 0$ (Very little attenuation of propagating wave, "dielectric" behaviour)

$n^2 \sim \epsilon_r$ is < 1 , $\oplus$ ve. $\rightarrow n < 1$ Phase velocity $> c$.

Group refractive index $n_g = \frac{d(\omega n)}{d\omega} = \frac{d}{d\omega} (\omega^2 - \omega_p^2)^{1/2} = \frac{\omega}{\sqrt{\omega^2 - \omega_p^2}}$

i.e. $n_g = (1 - \frac{\omega_p^2}{\omega^2})^{-1/2} < 1$.

This behaviour is seen in solids at soft X-ray wavelengths.

Application: pulsar signals in interstellar medium $N \sim 10^4/\text{m}^3$, $f_p \sim \text{few kHz}$
radio astronomy antennas detect pulses later at lower fregs.

Know distance $\rightarrow \omega_p \rightarrow N$ interstellar

Gives N interstellar $\rightarrow$ distance to pulsar

ii) Regime $\omega \sim \omega_p$.

$$\epsilon_r \sim 1 - \frac{\omega_p^2}{\omega^2 + \Gamma^2}$$

for $\omega^2 + \Gamma^2 \gg \omega_p^2$ $\epsilon_r \rightarrow \text{small}$, Γ finite.
Plasma behaves like a "metal", high reflection coeff

For $\omega^2 < \omega_p^2 - \Gamma^2$ ϵ_r is $\ominus$, n is imaginary, PURE ATTENUATION
NO PROPAGATION.

i.e. plasma behaves like a high-pass-filter $\omega > \sqrt{\omega_p^2 - \Gamma^2}$

Application: communications via ionosphere $N(h)$, $\omega_p \sim \text{few MHz}$ at several 100km above Earth. Γ typically a few Hz $\rightarrow$ kHz.

$\rightarrow$ radio wave at few MHz reflects at height where $\omega \sim \omega_p$.

$\rightarrow$ round-trip-curve-of-sight communication.

Ionospheric ionospheric diagnostics.

Application — optics of metals

conductivity measurements $\rightarrow \Gamma_c \sim 10^{13} \text{ Hz}$

$\gg \Gamma_r$

plasma frequencies $\sim 10^{15} \text{ Hz}$

Given general relation
$$n^2 = \frac{\mu\epsilon + i\mu\sigma/\omega}{\mu_0\epsilon_0}$$

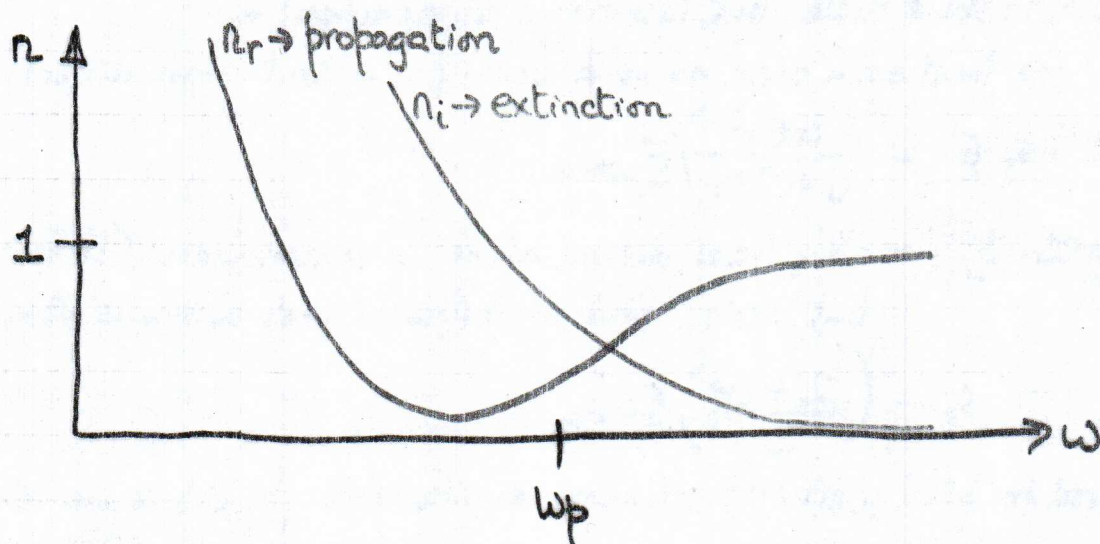
$$n^2 = \epsilon_r + i\sigma/\omega\epsilon_0 \quad \text{if } \mu_r = 1$$

Breaking down $n = n_r + in_i$ as before

$$n^2 = n_r^2 - n_i^2 + 2in_r n_i$$

So
$$n_r^2 - n_i^2 = 1 - \frac{\omega_p^2}{\omega^2 + \Gamma_c^2}$$

$$2n_r n_i = \frac{\omega_p^2 \Gamma_c}{\omega^2 + \Gamma_c^2}$$



At low freqs, metals are very opaque and highly reflective ($R \sim (\frac{1-n}{1+n})^2$)

just below plasma frequency $\rightarrow$ very lossy dielectric, $n_r < 1$

well above plasma frequency $\rightarrow$ "metals" become transparent dielectrics

isotropic situations, bound electrons.

Regard classical analogue of electron bound in atom/molecule in the i -th quantum state as bound harmonic oscillator, equivalent freq ω_{0i} , damping $m\Gamma_i$. (recall bound-scattering discussion earlier).

Then
$$m\ddot{x}_i + m\Gamma_i\dot{x}_i + m\omega_{0i}^2 x_i = qE$$

f_i = distribⁿ $f_i \rightarrow$ fraction of electrons in i 'th state

$$P = Ne \sum_i f_i x_i$$

$$\rightarrow P = \frac{Nq^2 E}{m} \sum_i \frac{f_i}{\omega_{0i}^2 - \omega^2 - i\omega\Gamma_i} = (\epsilon_r - 1)\epsilon_0 E$$

In solids, need to relate ϵ_{local} to $\epsilon_{\text{macroscopic}}$

$$= \epsilon_{\text{wave}} + \epsilon(\text{nearby atoms}) + \epsilon(\text{distant atoms})$$

Leave details to solid state, but two basic approaches:—

OSWALDER $\rightarrow$ treat each atom as in a cavity in a continuous dielectric

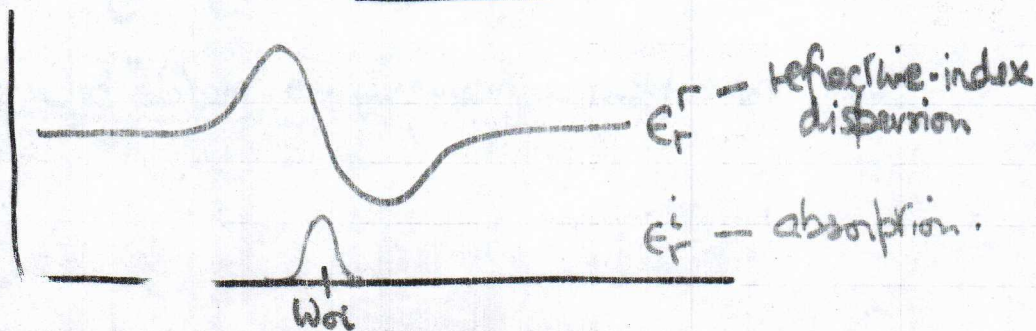
$$\rightarrow \epsilon = \left(\frac{3\epsilon_r}{1+2\epsilon_r} \right) \epsilon_{\text{wave}}$$

LORENTZ-LORENTZ $\rightarrow$ only treat distant atoms as a continuum (better approx) but neglect field distortions at each molecule/atom

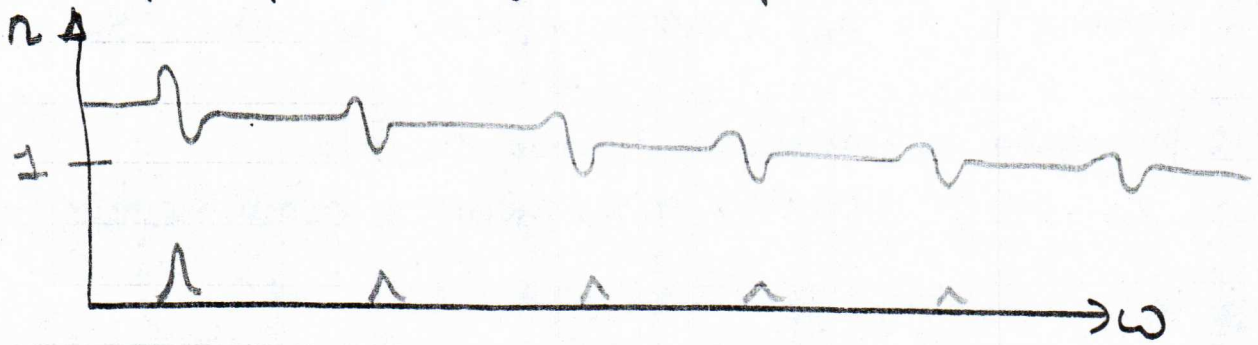
$$\rightarrow \epsilon = \left(\frac{2+\epsilon_r}{3} \right) \epsilon_{\text{wave}}$$

Oswalder good for polar materials, Lorentz-Lorentz good for dilute materials

Eikler way, ^{sign.} ~~excursions~~ excursions in $\epsilon_r(\omega)$ come when ω is near one of the ω_{0i} — then one term in $\sum_i$ dominates



→ "Anomalous" dispersion of materials near resonances ω_i where wave transfers much energy to bound electrons. Radiation ^{and collisional} damping then → large absorption of energy transferred from wave to electron motions.



i) at high freq., n of most materials → < 1 (X-ray).

ii) at low freq.

$$(\epsilon - 1) \rightarrow \sum_i \frac{f_i \omega_p^2}{\omega_{0i}^2 - \omega^2 - i\omega\gamma_i} \rightarrow \sum_i \frac{f_i \omega_p^2}{\omega_{0i}^2}$$

i.e. materials with many infrared ω_i s. (polar molecules) → large low-freq. n .

→ very high low-freq. dielectric constant of water.

Most glasses in visible are between resonances → slow $n(\omega)$ → PRISMS etc.
strongly absorbent in UV and IR.

(Note that we have derived this set of results from same model as was used earlier for bound-electron scattering, for a single $\omega_i = \omega_0$, $\gamma_i = \gamma$)

e.g. glass $N \sim \text{Atoms} \# / \text{cm}^3 \sim 6 \times 10^{24} / \text{cm}^3$ $\omega_0 \sim 2 \times 10^{16}$ (UV)
 $\gamma \sim 1$ $\omega \sim 4 \times 10^{15}$ (Visible)

$$n \approx \sqrt{1 + \frac{6 \times 10^{24} \times (1.6 \times 10^{-19})^2 \times f_{eff}}{(9.1 \times 10^{-31}) \times 8.85 \times 10^{-12} (4 \times 10^{15} - 16 \times 10^{16})}}$$

$$\sim \sqrt{1 + 4.7 f_{eff}} \sim 2 \quad (\text{reasonable!})$$

● Plasma with superimposed magnetic field $\underline{B}_0$

- magnetically biased dielectrics rotate plane of linear polarisation of light travelling $\parallel$ to $\underline{B}_0$ (Faraday effect), $\propto$ thickness, B_0
- radio propagation in ionosphere + geomagnetic field
interstellar ionised hydrogen + galactic magnetic fields.

Add $\underline{B}_0$ to eqn of motion of free electrons $\rightarrow \underline{F}_m = q \underline{\dot{r}} \times \underline{B}_0$

$$m \ddot{\underline{r}} = q(\underline{E} + \underline{\dot{r}} \times \underline{B}_0) - m \Gamma \underline{\dot{r}} \quad \text{now describes } \underline{r}(t)$$

Again take $\underline{P} = Nq\underline{r}$ and seek $\underline{P} = (\underline{\epsilon}_r - 1)\epsilon_0 \underline{E} \quad \underline{P} \sim e^{i\omega t}$

$$-m\omega^2 \underline{r} = q(\underline{E} + i\omega \underline{r} \times \underline{B}_0) - im\omega \Gamma \underline{r}$$

$$-m\omega^2 \underline{P} = Nq^2 \underline{E} + i\omega q \underline{P} \times \underline{B}_0 - im\omega \Gamma \underline{P}$$

$$\underline{P}(i\omega\Gamma - \omega^2) - i\omega \frac{q}{m} \underline{P} \times \underline{B}_0 = \frac{Nq^2}{m} \underline{E}$$

→ $\underline{\epsilon}_r$ must now have tensor character, $\underline{P} \cdot \underline{E}$ rel. depends on direction of $\underline{P}$ relative to $\underline{B}_0$.

Notation: $\frac{|q|B_0}{m} = \omega_g$

gyrofrequency/cyclotron frequency of electrons in superimposed $\underline{B}_0$, $\oplus$ by definition.

q is $\ominus$

so $\ominus i\omega \frac{q}{m} \underline{P} \times \underline{B}_0$ is $\oplus$ if $\underline{P} \times \underline{B}_0$ is $\oplus$

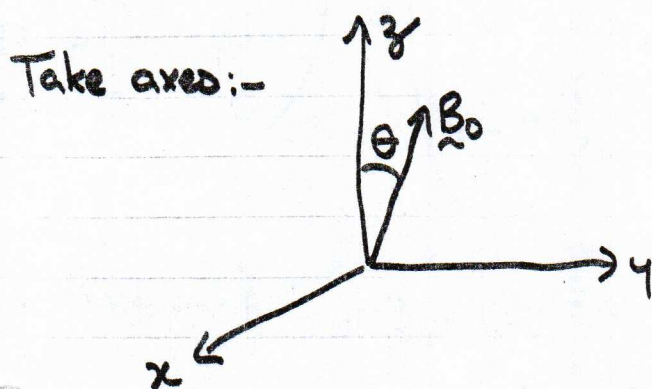
i.e. $\ominus i\omega \frac{q}{m} \underline{P} \times \underline{B}_0$ is $\equiv \oplus i\omega \omega_g \underline{P} \times \underline{a}_B$ ← unit vector parallel to $\underline{B}_0$

• We consider case $\Gamma \ll \omega, \omega_g$ (most practical cases will do this)

$$\Rightarrow -\omega^2 \underline{P} + i\omega\omega_g \underline{P} \times \underline{a}_B = \omega_p^2 \epsilon_0 \underline{E}$$

$$-\underline{P} = \frac{\omega_p^2}{\omega^2} \epsilon_0 \underline{E} - \frac{i\omega_g}{\omega} \underline{P} \times \underline{a}_B$$

Notation (common) $X = \omega_p^2/\omega^2$ $Y = \omega_g/\omega$ dimensionless parameters.



B_0 in zy plane
 B_0 at θ to z axis

$$\begin{aligned} -P_x &= X\epsilon_0 E_x - iY(P_y \cos\theta - P_z \sin\theta) \\ -P_y &= X\epsilon_0 E_y + iY(P_x \cos\theta) \\ -P_z &= X\epsilon_0 E_z - iY(P_x \sin\theta) \end{aligned}$$

These define the tensor constitutive relations for the magnetically-biased plasma. ~~These define the tensor constitutive relations for the magnetically-biased plasma.~~

After some manipulation, you would find:

$$\underline{\epsilon}_r = \begin{pmatrix} 1 + \frac{X}{Y^2 - 1} & \frac{iXY \cos\theta}{Y^2 - 1} & -\frac{iXY \sin\theta}{Y^2 - 1} \\ -\frac{iXY \cos\theta}{Y^2 - 1} & 1 + \frac{X - XY^2 \sin^2\theta}{Y^2 - 1} & \frac{XY^2 \sin\theta \cos\theta}{Y^2 - 1} \\ \frac{iXY \sin\theta}{Y^2 - 1} & \frac{XY^2 \sin\theta \cos\theta}{Y^2 - 1} & 1 + \frac{X - XY^2 \cos^2\theta}{Y^2 - 1} \end{pmatrix}$$

Note this is a diagonalisable hermitian matrix $\epsilon_{ij} = \epsilon_{ji}^*$ diagonalisable by a complex rotation of co-ordinates

General Property - Dielectric Tensor is Diagonalisable :

see Papers

The tensor will always be hermitian matrix if medium is lossless (collisionless)

e. we can change the basis of coordinates from $(\underline{a}_x, \underline{a}_y, \underline{a}_z)$ to another basis related by a complex rotation of coordinates and $\underline{\epsilon}$ will be diagonalisable. Representation of fields in the new basis will $\rightarrow$ characteristic plane wave refractive indices/polarisation combinations for anisotropic medium.

To see what this means, consider the case $\theta = 0^\circ$. $\cos\theta = 1$, $\sin\theta = 0$.

$$\underline{\epsilon}_r \Rightarrow \begin{pmatrix} 1 + \frac{x}{y^2-1} & \frac{ixy}{y^2-1} & 0 \\ -\frac{ixy}{y^2-1} & 1 + \frac{x}{y^2-1} & 0 \\ 0 & 0 & 1-x \end{pmatrix} = \begin{pmatrix} \text{general form} \\ a & ib & 0 \\ -ib & a & 0 \\ 0 & 0 & c \end{pmatrix}$$

If we transform to basis $\underline{a}_x + i\underline{a}_y$, $\underline{a}_x - i\underline{a}_y$, $\underline{a}_z$ then (check it!)

$$\begin{pmatrix} D_x + iD_y \\ D_x - iD_y \\ D_z \end{pmatrix} = \begin{pmatrix} a+b & 0 & 0 \\ 0 & a-b & 0 \\ 0 & 0 & c \end{pmatrix} \begin{pmatrix} E_x + iE_y \\ E_x - iE_y \\ E_z \end{pmatrix}$$

In the plasma case, this is:-

$$\begin{pmatrix} D_x + iD_y \\ D_x - iD_y \\ D_z \end{pmatrix} = \begin{pmatrix} 1 - \frac{x}{1-y} & 0 & 0 \\ 0 & 1 - \frac{x}{1+y} & 0 \\ 0 & 0 & 1-x \end{pmatrix} \begin{pmatrix} E_x + iE_y \\ E_x - iE_y \\ E_z \end{pmatrix}$$

This means that a wave with i) $\left. \begin{matrix} E_x + iE_y = A \\ E_x - iE_y = 0 \\ E_z = 0 \end{matrix} \right\}$ propagates with $n^2 = 1 - \frac{x}{1-y}$

$$\text{Recall } x = \frac{\omega_p^2}{\omega^2} \quad \omega_p^2 = \frac{Nq^2}{m\epsilon_0}$$

$$y = \frac{\omega_0}{\omega} \quad \omega_0 = \frac{gB_0}{m}$$

ii) $\left. \begin{matrix} E_x + iE_y = 0 \\ E_x - iE_y = A \\ E_z = 0 \end{matrix} \right\}$ propagates with $n^2 = 1 - \frac{x}{1+y}$

see (i) corresponds to

$$\underline{E}_y/\underline{E}_x = -i$$

LEFT CIRCULAR

rot of $\underline{E}$ with $\underline{a}$ at any pt

$$n^2 = 1 - \frac{x}{1-y}$$

(ii)

$$\underline{E}_y/\underline{E}_x = +i$$

RIGHT CIRCULAR



$$n^2 = 1 - \frac{x}{1+y}$$

Anisotropy without the tensors

Got the anisotropic case out by mathematical brute force.
Now let's get smarter

We know in any dielectric (no collisions, no σ)

$$\nabla \times \underline{H} = \underline{\dot{D}} \quad \nabla \times \underline{E} = -\underline{\dot{B}}$$

For monochromatic plane wave travelling // to z :-

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0 \quad \frac{\partial}{\partial t} = i\omega \quad \frac{\partial}{\partial z} = -ikn \quad \text{on } \underline{E}, \underline{H}$$

$$\begin{aligned} \text{so } ikn H_y &= i\omega D_x & ikn E_y &= -i\omega B_x \\ -ikn H_x &= i\omega D_y & -ikn E_x &= -i\omega B_y \end{aligned}$$

$$\text{i.e. } \frac{H_y}{H_x} = -\frac{D_x}{D_y} \quad \text{and} \quad \frac{B_y}{B_x} = -\frac{E_x}{E_y}$$

But if μ is scalar $\frac{H_y}{H_x} = \frac{B_y}{B_x} \therefore \frac{D_y}{D_x} \text{ must} = \frac{E_y}{E_x}$ even if the ϵ_r is tensor!

i.e. IF we have wave propagating with unique n in z direction

$$\frac{D_y}{D_x} \text{ must} = \frac{E_y}{E_x} = r \text{ (say)} \leftarrow \text{characterises wave. pol.}$$

This means also that $\frac{P_y}{P_x} \text{ must} = r$ (check it)

i.e. $\boxed{\frac{P_y}{E_x} = \frac{P_x}{E_y}}$ in field combinations for which n is unique

USE THIS TO FIND CHARACTERISTIC WAVES

Note also 2-cpls of curl eqns $\rightarrow \left. \begin{aligned} D_z &= 0 \\ B_z &= 0 \end{aligned} \right\}$ Waves are transverse
in $\underline{D}, \underline{B}$ in general

(not $\underline{E}, \underline{H}$ - this is only
in an isotropic med.)

Note that in a dielectric $\nabla \times \underline{H} = \underline{\dot{D}}$ $\nabla \times \underline{E} = -\underline{\dot{B}}$ everywhere

monochromatic (ω)
 For plane waves travelling $\parallel$ to z , $\frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$ (def? of plane wave)
 $\frac{\partial}{\partial t} = i\omega$ $\frac{\partial}{\partial z} = -ikz$

Then these eqns $\rightarrow$

$$\begin{aligned} ikz H_y &= i\omega D_x \\ -ikz H_x &= i\omega D_y \end{aligned}$$

$$\begin{aligned} ikz E_y &= -i\omega B_x \\ -ikz E_x &= -i\omega B_y \end{aligned}$$

$$0 = i\omega D_z$$

$$0 = -i\omega B_z$$

1) $\rightarrow$ Plane waves are transverse in $\underline{D}, \underline{B}$ (not $\underline{E}, \underline{H}$)
 i.e. if there is a longitudinal field E_z it must come entirely from dielectric polarisation, so that $P_z = -\epsilon_0 E_z$ and $D_z = 0$.

2) $\rightarrow \frac{H_y}{H_x} = -\frac{D_x}{D_y} \quad \frac{E_y}{E_x} = -\frac{B_x}{B_y} = (\mu_r = 1) - \frac{H_x}{H_y} = \frac{D_y}{D_x}$ from preceding

i.e. the ratio $\frac{D_y}{D_x}$ must = ratio $\frac{E_y}{E_x}$ in plane wave $\parallel z$, refr. ind.

~~In an isotropic medium this will only be possible for certain E_y/E_x combos.~~
 This applies even if the dielectric is anisotropic, if the wave propagates with a unique refractive index n .

In an anisotropic medium this will only be possible for certain E_y/E_x combos, i.e. the medium must behave as if the dielectric tensor were diagonal

i.e. $\begin{pmatrix} n^2 & 0 & 0 \\ 0 & n^2 & 0 \\ 0 & 0 & ? \end{pmatrix}$ for $\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$ in the normal coordinates.

If the dielectric tensor has off-diagonal elements in the normal coordinate system, we must seek alternate representations in which it is diagonal. Representing the fields E_x, E_y, E_z in the diagonalising coordinate system will tell us what combinations of E_x, E_y, E_z separately (i.e. linear polarisations) can propagate as plane waves with a unique refractive index in our particular medium.

Propagation $\parallel$ to $\underline{B}_0$.

$$-P_z = X\epsilon_0 E_x - iY P_y \quad (1)$$

$$-P_y = X\epsilon_0 E_y + iY P_x \quad (2)$$

$$-P_z = X\epsilon_0 E_z \quad \text{and} \quad D_z = 0 \text{ requires } P_z = -\epsilon_0 E_z$$

Hence $P_z = E_z = 0$ is only solution in general ω .

$$\text{Put } \frac{E_y}{E_x} = r$$

$$\text{Then } \frac{-P_x}{E_x} = \epsilon_0 X - iY \frac{P_y}{E_x} \quad \text{from (1)/} E_x$$

$$\frac{-P_y}{E_x} = \epsilon_0 X r + \frac{iY P_x}{E_x} \quad \text{from (2)/} E_x$$

$$\begin{aligned} \frac{-P_x}{E_x} &= \epsilon_0 X + iY \left(\epsilon_0 X r + \frac{iY P_x}{E_x} \right) \\ &= \epsilon_0 X (1 + iY r) - Y^2 \left(\frac{P_x}{E_x} \right) \end{aligned}$$

$$\therefore \frac{P_x}{E_x} = \epsilon_0 \left(\frac{X + iXYr}{Y^2 - 1} \right)$$

$$\text{Similarly } \frac{P_y}{E_y} = \epsilon_0 \left(\frac{X - iXY/r}{Y^2 - 1} \right)$$

$$\frac{P_x}{E_x} = \frac{P_y}{E_y} \quad \text{iff} \quad iXYr = -\frac{iXY}{r}$$

$$\text{i.e. } r = \pm i$$

$$\text{So } \frac{P_x}{E_x} = \frac{P_y}{E_y} = (R_r^2 - 1)\epsilon_0 = \frac{X \mp XY}{Y^2 - 1} = -\frac{X}{1 \pm Y} \text{ as before}$$

Faraday Effect

Circular pol: $\frac{E_y}{E_x} = +i$, complex "right" amplitude R $\underline{E}_R = (R\hat{a}_x + iR\hat{a}_y)e^{i\omega t}$
 $\frac{E_y}{E_x} = -i$, complex "left" amplitude L $\underline{E}_L = (L\hat{a}_x - iL\hat{a}_y)e^{i\omega t}$

Add them $\underline{E} = \underline{E}_R + \underline{E}_L = \{(R+L)\hat{a}_x + i(R-L)\hat{a}_y\}e^{i\omega t}$

$R=L \rightarrow$ lin pol $\parallel x$

$R=-L \rightarrow$ lin pol $\parallel y$

General case $|R|=|L| \rightarrow R = Le^{i\varphi}$ i.e. R leads L by φ .

$\underline{E} = L\{(1+e^{i\varphi})\hat{a}_x + i(e^{i\varphi}-1)\hat{a}_y\}e^{i\omega t}$

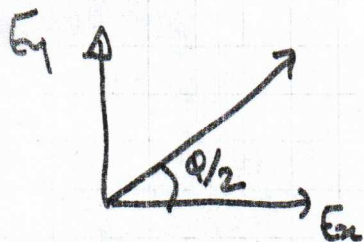
$= L\{(e^{-i\varphi/2} + e^{i\varphi/2})\hat{a}_x + i(e^{i\varphi/2} - e^{-i\varphi/2})\}e^{i\omega(t+\varphi/2)}$

$\rightarrow \frac{E_y}{E_x} = \frac{i(e^{i\varphi/2} - e^{-i\varphi/2})}{e^{i\varphi/2} + e^{-i\varphi/2}} = \frac{i(2i\sin\varphi/2)}{2\cos\varphi/2}$

$\cos = \frac{e^{ix} + e^{-ix}}{2}$

$\sin = \frac{e^{ix} - e^{-ix}}{2i}$

$= -\tan\varphi/2$



i.e. Phase difference of φ between R & L
rotates lin pol. sum through $\varphi/2$.

In plasma at $\omega > \omega_p$

Right circular $n_+^2 = 1 - \frac{X}{1+Y}$

Left circular $n_-^2 = 1 - \frac{X}{1-Y}$

$n_+^2 - n_-^2 = (n_+ + n_-)(n_+ - n_-)$

$\sim 2(n_+ - n_-)$

$= \frac{X}{1-Y} - \frac{X}{1+Y} = \frac{X+XY - X+XY}{1-Y^2}$

$\therefore n_+ - n_- = \frac{XY}{1-Y^2} = \frac{\omega_p^2 \omega_g}{\omega(\omega^2 - \omega_g^2)} \rightarrow \frac{\omega_p^2 \omega_g}{\omega^3}$

i.e. phase difference after propagation through path d

$$\phi = \frac{2\pi(n_+ - n_-)d}{\lambda} = \frac{2\pi\omega_p^2 \omega_q d}{\lambda \omega^3} = \frac{Nq^2}{m\epsilon_0} \frac{qB}{m} d \cdot \frac{1}{c\omega^2}$$

Rotation of plane of polarisation $\oplus$ leads $\ominus$

$$\text{is } \Delta\theta = \frac{1}{2}\phi = \frac{Nq^3 B d}{2m^2 \epsilon_0 c \omega^2}$$

$$= 9.34 \times 10^5 \frac{NBd}{\omega^2} \text{ (mts units)}$$

e.g. 40 MHz in ionosphere ~ 10 radians!!

$$\begin{aligned} N &\sim 10^{18} \\ B &\sim 3 \times 10^{-5} \\ d &\sim 3 \times 10^5 \end{aligned}$$

$$NBd \sim 10^{12} \text{ (variable)}$$

$$\Delta\theta \sim \frac{10^{18}}{\omega^2} \text{ radians}$$

large below $\omega \sim 10^9$ (few 100 MHz)

Lab Faraday effect

$$\Delta\theta = V B d$$

$$L \text{ "Verder" Constant} = \frac{Nq^3}{2m^2 \epsilon_0 c \omega^2}$$

$$V = \frac{\pi \omega_p^2}{\lambda \omega^3} \left(\frac{q}{m} \right)$$

In a dielectric with one ω_0 .

$$V = \frac{\pi \omega_p^2}{\lambda} \left(\frac{q}{m} \right) \left(\frac{\omega}{(\omega_0^2 - \omega^2)^2} \right)$$

more realistically

$$V = \frac{\pi \omega_p^2}{\lambda} \left(\frac{q}{m} \right) \omega \sum_i \frac{f_i}{(\omega_{0i}^2 - \omega^2)^2}$$

Near an $\omega_0 \rightarrow$ large birefringence ^{under B}

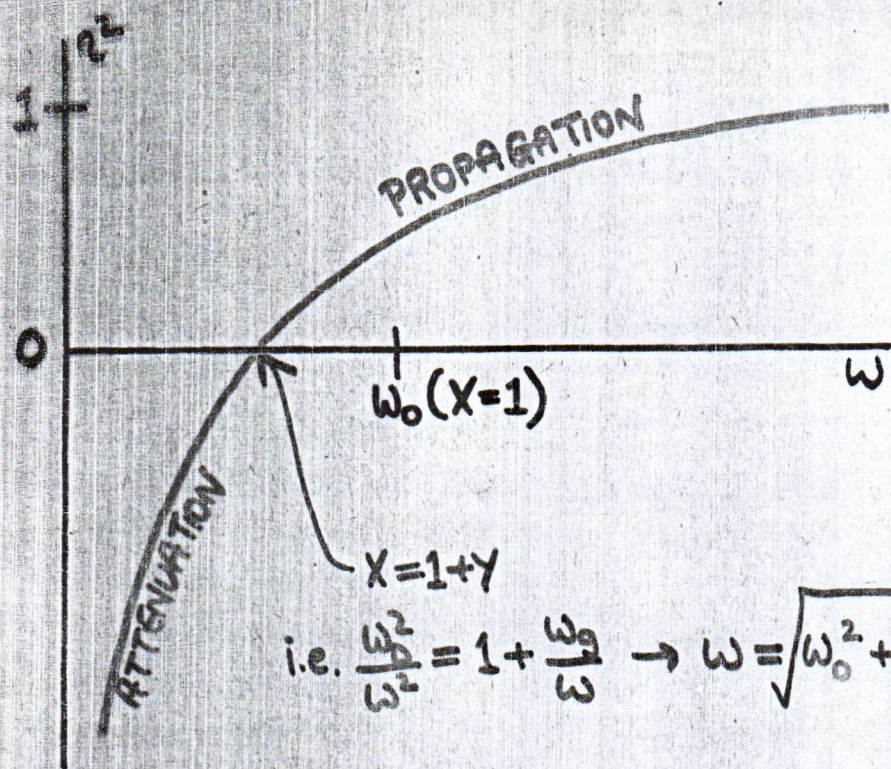
Seen in gases optically $\rightarrow$ VOIGT EFFECT.

Natural rotators ("optical activity")

$$\theta = \delta l$$

- e.g. quartz crystals
sugars.

$\uparrow$ "specific rotatory power"



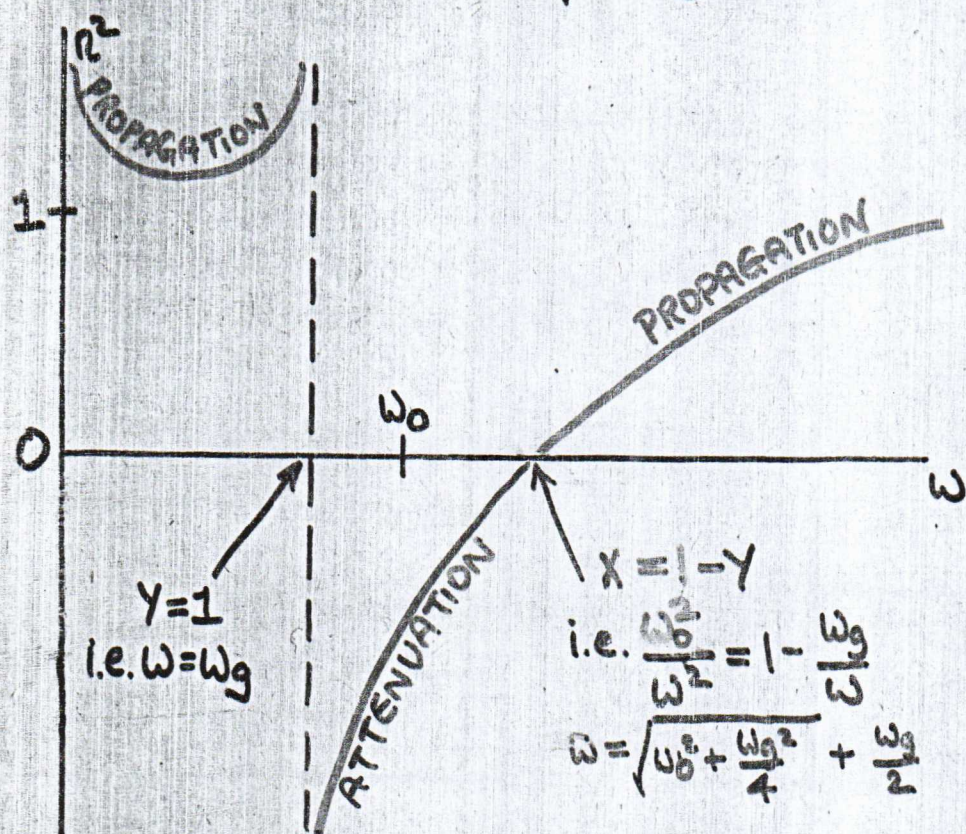
$$\tau = +i$$

$$n^2 = 1 - \frac{X}{1+Y}$$

$n^2 \rightarrow 0$ at frequency below plasma oscillation frequency ω_0 .

$$\text{i.e. } \frac{\omega_0^2}{\omega^2} = 1 + \frac{\omega_g}{\omega} \rightarrow \omega = \sqrt{\omega_0^2 + \frac{\omega_g^2}{4}} - \frac{\omega_g}{2}$$

This n^2 - ω curve resembles that for zero-field, displaced to lower ω .
 → **ORDINARY** mode of propagation (terminology)



$$\tau = -i$$

$$n^2 = 1 - \frac{X}{1-Y}$$

$n^2 \rightarrow 0$ at frequency above plasma frequency ω_0 .

$$n^2 \rightarrow \infty \text{ at } \omega = \omega_g$$

AND

New propagation regime ($n^2 > 1$) for $0 < \omega < \omega_g$

This n^2 - ω has qualitatively new phenomena at $\omega \leq \omega_g$
 → **EXTRAORDINARY** mode of propagation.

Group refractive index for extraordinary wave at $\omega \ll \omega_g$

$$n_g = \frac{d(\omega n)}{d\omega} = \frac{\omega_0}{2\sqrt{\omega\omega_g}}$$

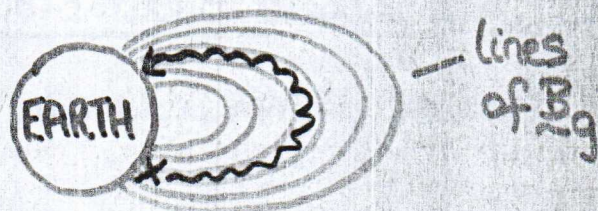
$\therefore$ Group velocity of pulse centred on frequency $\omega = \frac{2c\sqrt{\omega\omega_g}}{\omega_0}$

Detailed treatment shows that group velocity is greatest in directions parallel to $\underline{B}_g$, so pulses are "guided" by $\underline{B}_g$.

lightning flash in one hemisphere (x) generates wide-band radio pulse.

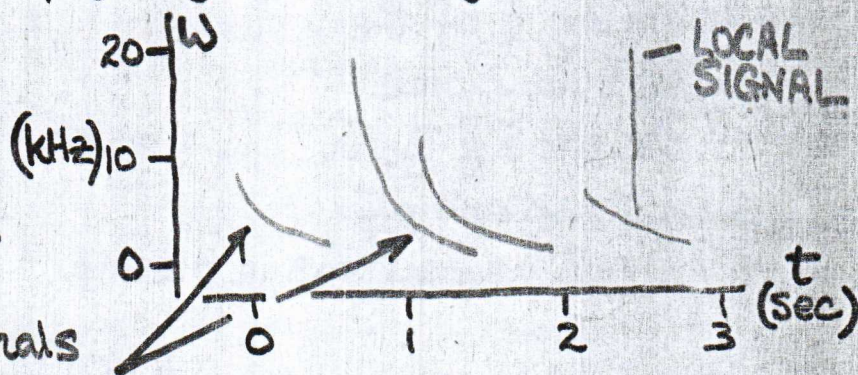
Audio-frequency receivers in other hemisphere receive guided

extraordinary wave pulse propagating with velocity $\propto \sqrt{\omega}$



Tracing of dynamic spectrum of radio noise received by low-frequency receiver:—

"Whistler" signals



Shape of whistler spectrum depends on distribution of ω_g and N along whistler's path $\rightarrow$ tool for studying $N(h)$ and $\underline{B}_g(h)$ in upper ionosphere using ground-based equipment. -e.g. Helliwell, "Whistlers and Related Ionospheric Phenomena", Stanford Univ. Press (1965)

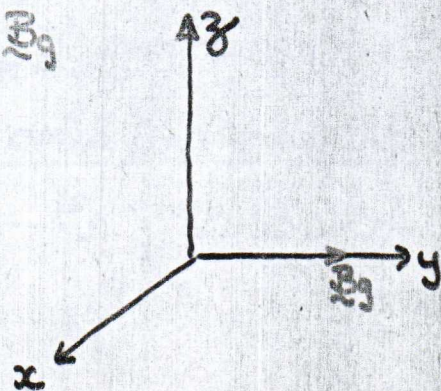
b) Case $\alpha = 90^\circ$. Propagation perpendicular to $\underline{B}_g$

$$\text{Eqns. 64} \rightarrow -P_x = \epsilon_0 X E_x + iY P_z$$

$$-P_y = \epsilon_0 X E_y$$

$$-P_z = \epsilon_0 X E_z - iY P_x$$

$\underline{E} = E_y \underline{a}_y$ does not $\rightarrow$ electron velocities $\perp$ to $\underline{B}_g$. Hence no $\underline{v} \times \underline{B}_g$ forces if $\tau = \infty$.



1. Large reflection coefficients in three frequency regimes

$$\Gamma \rightarrow 1 \text{ as } n \rightarrow 0 \text{ or } \infty$$

i.e. ionosphere will become highly reflective // to B_0 if

i) $\omega = \omega_g$ $r = -i$ component

ii) $\omega = \sqrt{\omega_0^2 + \omega_g^2/4} + \omega_g/2$ $r = -i$ component

iii) $\omega = \sqrt{\omega_0^2 + \omega_g^2/4} - \omega_g/2$ $r = +i$ component

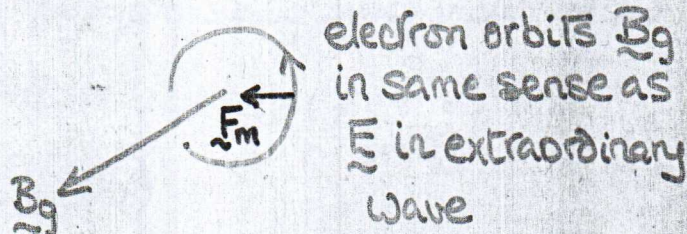
2. What happens To extraordinary wave at $\omega = \omega_g$? ($n^2 \rightarrow \pm \infty$)

Consider electron motions in plasma

$\vec{v}$
 $\rightarrow \vec{v} \times \vec{B}_0$
 $F_m = e\vec{v} \times \vec{B}_0$ (e $\ominus$ ve)

At $\omega = \omega_g$, $\vec{E}$ of the extraordinary wave rotates around $\vec{B}_0$ in same sense and at

same rate as electrons in plasma.



Radius of electron orbit $r = \frac{m\vec{v}}{eB_0}$

continuously increased as electrons accelerated by $\vec{E}$ of the wave

Electron orbits become indefinitely large (in absence of collisions) and energy of wave is absorbed and stored as orbital energy of the electrons.

In ordinary wave, $\vec{E}$ rotates in wrong direction to do this at $\omega = \omega_g$

3. Whistlers and atmospherics

At $\omega \ll \omega_g$ $n^2 \rightarrow 1 + \frac{\omega_p^2}{\omega \omega_g} \rightarrow \frac{\omega_p^2}{\omega \omega_g}$ (extraordinary wave only)

Physics 831* Exercise 6

Collisionless plasma eqn. of motion $m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B}_0)$

$$\rightarrow -\mathbf{P} = X\epsilon_0\mathbf{E} - iY(\mathbf{P} \times \hat{\mathbf{n}}_B)$$

When $\hat{\mathbf{n}}_B = \hat{\mathbf{y}}$ ($\mathbf{B}_0 \parallel y$ axis)

$$-P_x = X\epsilon_0 E_x + iY P_z$$

$$-P_y = X\epsilon_0 E_y$$

$$-P_z = X\epsilon_0 E_z - iY P_x$$

For propagation $\parallel$ to z , look for $\frac{P_x}{E_x} = \frac{P_y}{E_y}$ condition

also satisfying $D_z = 0$, i.e. $-P_z = \epsilon_0 E_z$

Note that $\frac{P_y}{E_y} = -X\epsilon_0$ regardless of E_x

and $-P_z = -XP_z - iYP_x$, i.e. $P_z = \frac{iYP_x}{1-X}$ regardless of E_y, P_y

$$\text{so } -P_x = X\epsilon_0 E_x - \frac{Y^2}{1-X} P_x$$

$$\text{i.e. } \frac{P_x}{E_x} = -\epsilon_0 X / (1 - Y^2/(1-X))$$

regardless of E_y, P_y

The conditions are decoupled, and the only way to satisfy them both is if either $E_y = 0$, when $n^2 = 1 - \frac{X}{1 - Y^2/(1-X)}$

$$\text{i.e. } n^2 = 1 - \frac{X(1-X)}{1-X-Y^2}$$

$$\text{or } E_x = 0 \text{ when } n^2 = 1 - X$$

i.e. Cherenkov waves are linear polarisations.

Ordinary mode Lin pol. $\parallel \mathbf{B}_0$ ($E_x = 0$) $n^2 = 1 - X$ doesn't know $\mathbf{B}_0$ is there

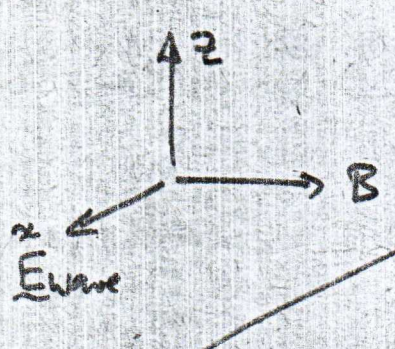
Extraordinary mode Lin pol. $\perp \mathbf{B}_0, \hat{\mathbf{z}}$ ($E_y = 0$) $n^2 = 1 - \frac{X(1-X)}{1-X-Y^2}$

Extraordinary mode we showed $P_z = \frac{i\gamma P_x}{1-x}$

$$\text{i.e. } z = \frac{i\gamma x}{1-x}$$

→ ellipse in z - x plane, axial ratio $\frac{z}{x} = \frac{\gamma}{1-x}$.

When $\omega = \omega_p$, $x = 1$, we have linear s.h.m. $\parallel$ to z axis.
no motion $\parallel$ to x .



$\ddot{m}\ddot{x} = qEx - q\dot{z}B_0$
 $m\ddot{z} = qE_z + q\dot{x}B_0$
But $E_z = -\frac{Nqz}{\epsilon_0}$

So $m\ddot{z} = -$

$$\ddot{z} = \frac{-\omega\gamma x}{1-x} = \frac{-qB_0 x}{m(1-x)} \stackrel{\text{also}}{=} -\frac{Nq^2 x}{m\epsilon_0(1-x)}$$

$$\rightarrow m\omega_p^2 x = qEx - q i\omega z B_0$$

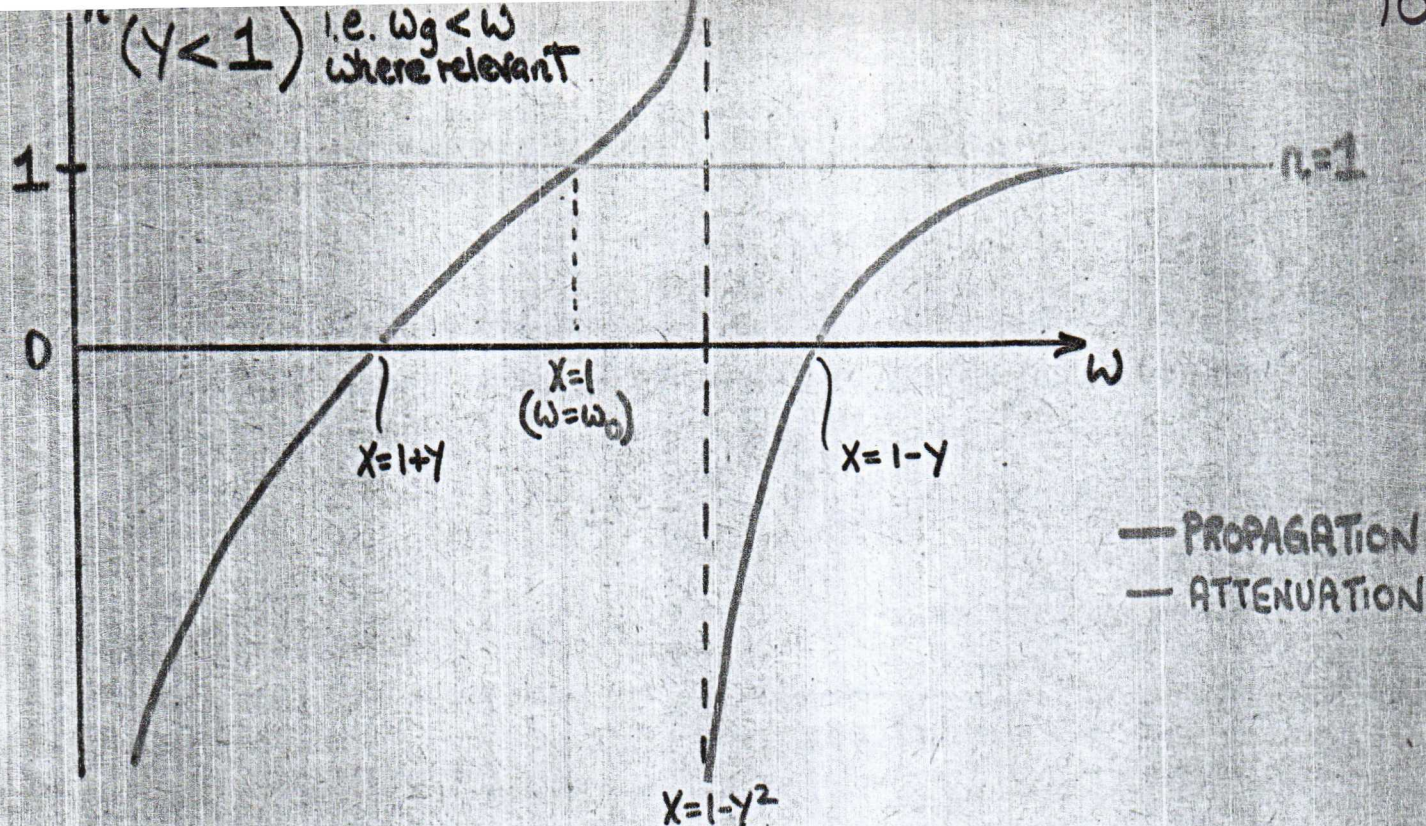
$$\rightarrow m\omega_p^2 z = qE_z + q i\omega x B_0$$

$$\text{But } E_z = -\frac{Nez}{\epsilon_0} = -m\omega_p^2 z$$

$$\text{i.e. } x = 0.$$

$$\text{i.e. } \underline{qEx = q i\omega z B_0.}$$

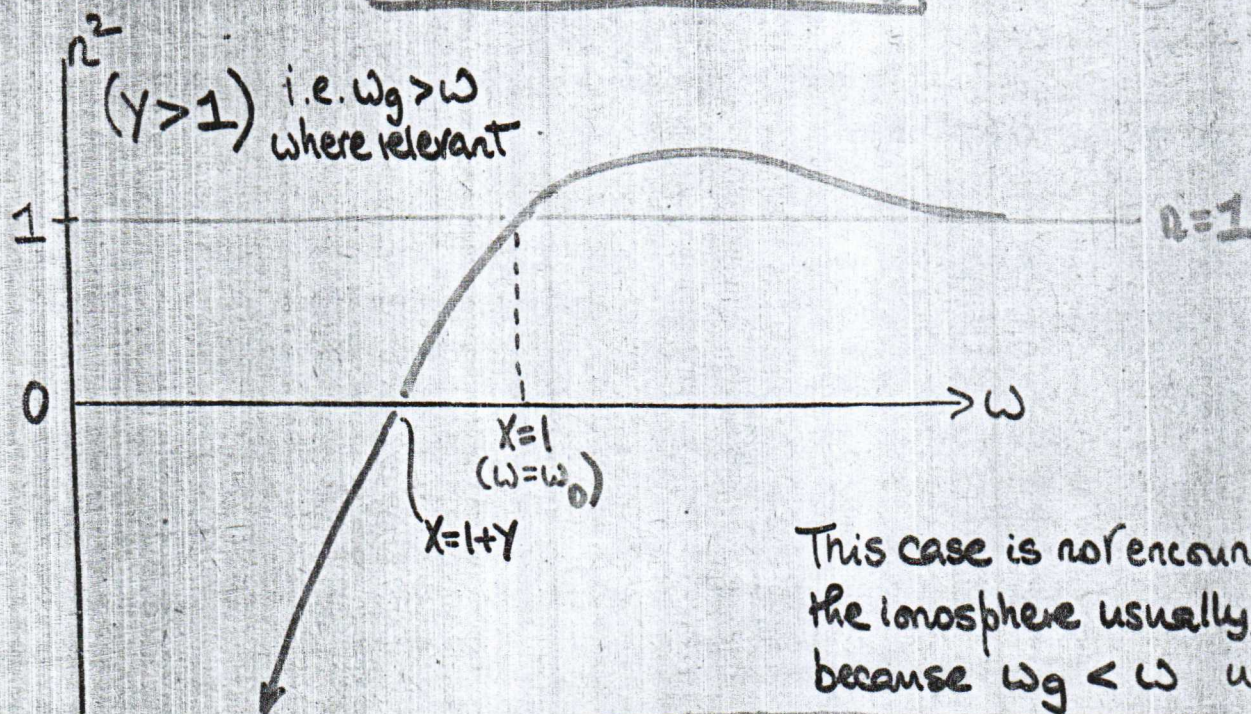
Electric and magnetic forces $\parallel$ to x are cancelling always.



Now two pass-bands $X < 1-\gamma$, i.e. $\omega > \sqrt{\omega_0^2 + \omega_g^2/4} + \omega_g/2$ $n < 1$

$1-\gamma^2 < X < 1+\gamma$ i.e. $\sqrt{\omega_0^2 + \omega_g^2/4} - \omega_g/2 < \omega < \sqrt{\omega_0^2 + \omega_g^2}$

$n \geq 1$ as $\omega \geq \omega_0$



This case is not encountered in the ionosphere usually, because $\omega_g < \omega$ usually.

One pass band only $X < 1+\gamma$ $n \geq 1$ as $\omega \geq \omega_0$

Similarly $X > 1-\gamma$

Look for $B_y/B_x = \omega_g/\omega$

hence

SUMMARY OF PHENOMENA — B_0 transverse.

Refractive index no longer $\rightarrow 0$ at unique frequency.

Ordinary and extraordinary waves are linearly polarised.

Ordinary wave is unaffected by B_0 .

THINK: What happens if we try to send linear pol. at 45° to ordinary wave, along direction transverse to B_0 ?
What happens if we try to send circular polarisation transverse to B_0 ?

General case. ($0^\circ < \alpha < 90^\circ$)

$$n^2 = 1 - \frac{X}{1 + i\gamma \cos \alpha}$$

$$r = \frac{E_y}{E_x} = -\frac{i}{2\omega_0 \cos \alpha} \left(\frac{\omega_0^2 \sin^2 \alpha}{1 - X} \pm \sqrt{\left(\frac{\omega_0^2 \sin^2 \alpha}{1 - X} \right)^2 + 4\omega_0^2 \cos^2 \alpha} \right)$$

(normal modes are elliptically polarised)

In practice, must add collisions

ion motions

variation of B_0 with height

variation of ω_0 with height

to get "proper" description.

Good, short introduction to ionospheric physics:

J.A. RATCLIFFE, "An Introduction to the Ionosphere and Magnetosphere"

(Cambridge University Press 1972) QC879. R34

Ionosphere Bible

K.G. BUDDEN, "Radio Wave Propagation in the Ionosphere"

(Cambridge University Press 1961) QC973. B92

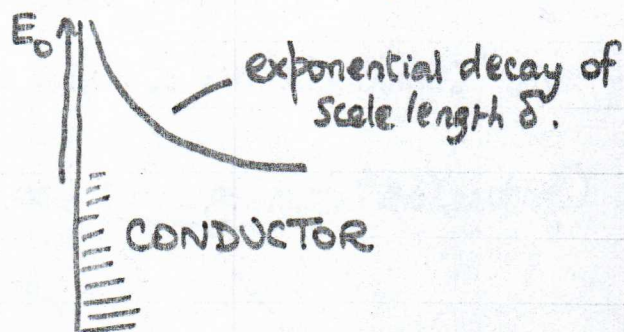
Ch. 2-4 are useful general review of em wave concepts.

Guided Waves

Boundary Conditions at good conductors :

— Apply $E_{\parallel} = E_{\parallel 0} e^{i\omega t}$ parallel to plane boundary of a good conductor: -

$$J_{\parallel} = \sigma E_{\parallel} \text{ is limited.}$$



In limit $\sigma \rightarrow \infty$
 $\delta = \sqrt{\frac{2}{\omega \mu \sigma}} \rightarrow 0$

i.e. MACROSCOPICALLY, $E_{\parallel} \rightarrow 0$ "at surface" in conductor.
 $\underline{n} \times \underline{E} \rightarrow 0$.

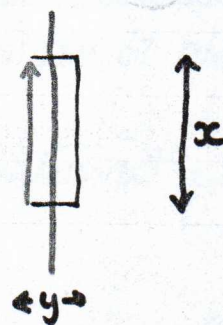
But because $\nabla \times \underline{E} = -\dot{\underline{B}}$ in any medium, this also means that $\underline{n} \times \underline{E} \rightarrow 0$ above the conductor in the medium of incidence.

$$\nabla \times \underline{E} = -\dot{\underline{B}} \Rightarrow \int_{\Gamma} \underline{E} \cdot d\underline{\ell} = - \iint_S \dot{\underline{B}} \cdot d\underline{S}$$

In diagram let $y \rightarrow 0$

$$\int_{\Gamma} = (E_{\parallel, \text{med}} - E_{\parallel, \text{cond}}) x$$

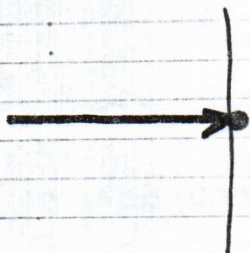
$$\iint_S = -\dot{B} x y \rightarrow 0$$



$$\therefore E_{\parallel, \text{med}} \Rightarrow E_{\parallel, \text{cond}} \Rightarrow 0 \text{ as } \sigma \rightarrow \infty.$$

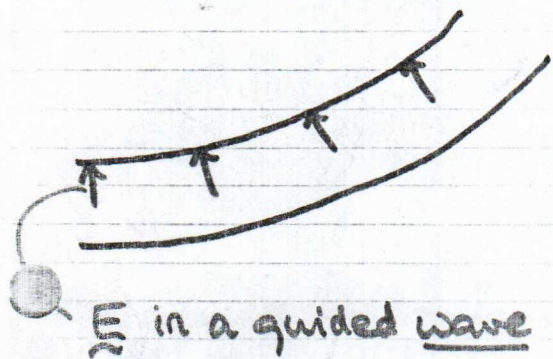
In the medium, this means $E_{\parallel, i}$ must be cancelled by reflected $E_{\parallel, r}$ in antiphase to satisfy boundary conditions $\rightarrow$ "good reflector".

$E_{\perp}$ can terminate on a charge at the surface.



$H_{\parallel}$ can "terminate" on a line current at surface.

→ Conductors can guide wave propagation



Reflections will adjust to keep $\underline{n} \times \underline{E} = 0$ everywhere on the surface.

Curved surface → curved propagation will also modify n in direction of prop. by interference $n \rightarrow \hat{n}$.

$\underline{E}$ in a guided wave (at the conducting surface).

Treatment of Guided Waves (first approx.)

→ Solutions to vector Helmholtz Eqns in the medium of propagation (usually a dielectric)

satisfying boundary conditions $\underline{n} \times \underline{E} = 0$ at all conducting surfaces.

Slight modification for real metals, not superconductors. Calculate wave fields as if $\sigma \rightarrow \infty$. Give $\underline{E}$, $\underline{H}$ distribⁿ on conductors.

$\Delta \underline{H}_{\parallel} \rightarrow$ surface currents $\underline{J}$. Calculate power loss in these due to finite σ , assuming the "skin-depth" $\underline{E}$, $\underline{J}$ distribⁿ.

Conditions for Guided TEM waves

Propagation Medium — Permeable Dielectric $\mu = \mu_r \mu_0$ $\epsilon = \epsilon_r \epsilon_0$.

Consider propagation || to z , All E cpts follow $E_0 e^{i\omega(t - \hat{n}z/c)}$
 $\hat{n}$ = guided refractive index (modified by interference)

$$\text{guided phase velocity} = c/\hat{n}$$

(In TEM mode by definition $E_z = H_z = 0$)

Wave field $\underline{E}$ satisfies vector Helmholtz Eqn ($k_0 = \omega/c$)

$$\boxed{(\nabla^2 + k_{\text{guide}}^2) \underline{E} = 0}, \text{ i.e. } \boxed{(\nabla^2 + k_0^2 n_d^2) \underline{E} = 0}$$

n_d = free-propagation ref. index of medium.

But $\frac{\partial}{\partial z} \equiv -i\omega \frac{\hat{n}}{c} \times \equiv -ik_0 \hat{n}$ due to imposition of metallic boundary cond^{ns} and reflections.

i.e. wave field satisfies $(\nabla_t^2 + k_0^2 (n_d^2 - \hat{n}^2)) \underline{E} = 0$
 $\uparrow$ transverse Laplacian

This is true for any guided wave, whether TEM or not. Also $\underline{H}$ satisfies same equation for any guided wave.

If wave is TEM,
 $E_z = 0, H_z = 0$

$$\nabla \times \underline{E} = -\dot{\underline{B}}$$

$$\begin{array}{l} \text{x-cpt} \rightarrow \frac{\hat{n} E_y}{c} = -\mu H_x \\ \text{y-cpt} \rightarrow \frac{\hat{n} E_x}{c} = +\mu H_y \end{array}$$

$$\nabla \times \underline{H} = -\dot{\underline{D}} \quad (\text{no } \underline{J} \text{ in medium of propagation})$$

$$\begin{array}{l} \text{x-cpt} \rightarrow \frac{\hat{n} H_y}{c} = \epsilon E_x \\ \text{y-cpt} \rightarrow \frac{\hat{n} H_x}{c} = -\epsilon E_y \end{array}$$

$$\begin{array}{l} \hat{n}^2 = c^2 \mu \epsilon \\ \hat{n} = n_d \end{array}$$

$\rightarrow$ i.e. $\hat{n} = n_d$ for TEM mode even when it is guided.

- the phase velocity of a GUIDED TEM MODE is exactly that of the unguided mode in the same material. Only difference is the collimated power flow produced by the wave confinement due to guiding conductors.

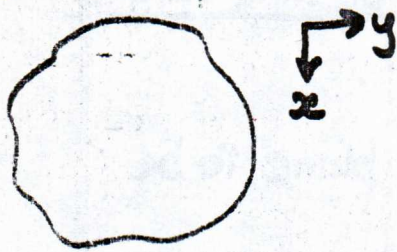
But $\underline{E}$ must satisfy
$$\left[\nabla_t^2 + k_0^2 (n_d^2 - \hat{n}^2) \right] \underline{E}^H = 0$$

If $n_d = \hat{n}$ in TEM, so:
$$\boxed{\nabla_t^2 \underline{E}^H = 0}$$

i.e. the instantaneous fields at any z satisfy Laplace's Eqn. (as if $\nabla \times \underline{E} = 0$ and no contribution from $\underline{B}$).

Instantaneous $\underline{E}$ field in a guided TEM mode at given z must be an electrostatic field configuration.

115
Surfaces of guiding conductors must therefore be equipotentials in a given xy plane when a TEM wave mode propagates between them || 3.



Consider hollow conductor of arbitrary cross-section in xy plane. Electrostatic potential constant over surface, must also satisfy $\nabla^2 \phi = 0$ inside conductor (no free charges in the dielectric there).

Now ϕ cannot have a maximum or minimum in the cavity.

If it had a maximum $\frac{\partial^2 \phi}{\partial x^2}$ and $\frac{\partial^2 \phi}{\partial y^2} < 0$ then

$$\therefore \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \neq 0$$

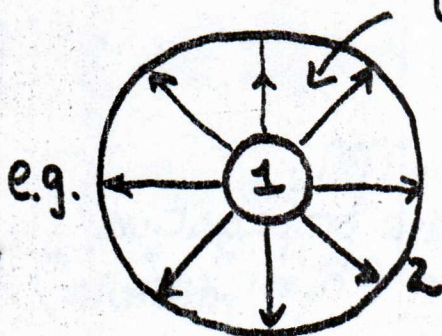
(Similar argument for minimum).

$\therefore$ If ϕ is constant over closed confining surface in xy plane it is constant everywhere within that surface.

$\therefore \underline{E} = -\underline{\nabla} \phi$ is zero everywhere within that surface.

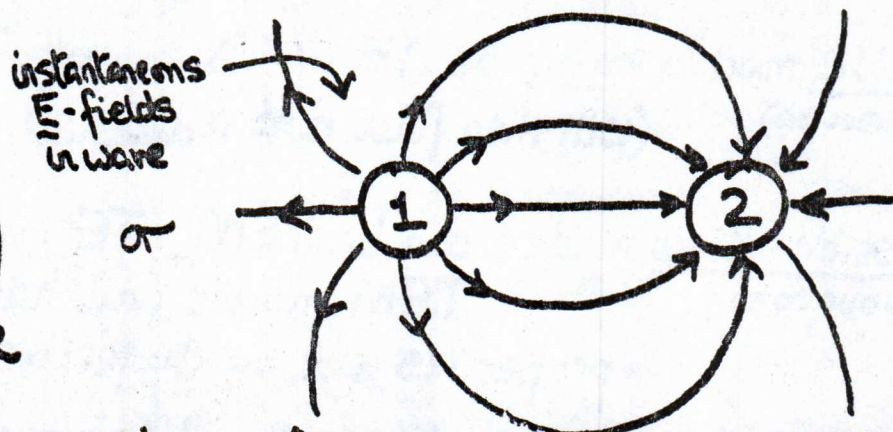
$\therefore$ TEM mode is impossible inside hollow conductor.

This situation resolved if more than one conducting surface involved.



e.g.

COAXIAL TRANSMISSION LINE



'OPEN-WIRE' TRANSMISSION LINE

Interpretation in terms of $\underline{H}$ in TEM mode.

$$\begin{aligned}\nabla \times \underline{H} &= \underline{J} + \underline{\dot{D}} \\ \nabla \cdot \underline{E} &= 0\end{aligned}$$

→ lines of $\underline{H}$ are closed loops surrounding either conduction currents, or time-varying lines of $\underline{D}$

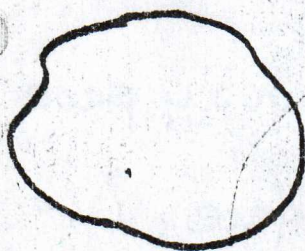
In general TEM wave $\underline{H}$ loops must lie in xy plane to be transverse to direction of propagation (z)

∴ the $\underline{J}$ or $\underline{\dot{D}}$ producing $\underline{H}$ in TEM wave must be $\parallel$ to z.

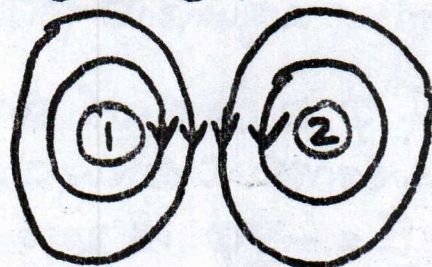
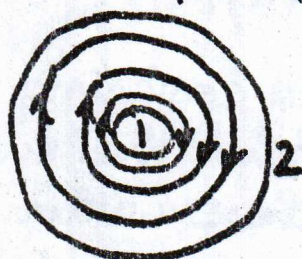
As $\underline{\dot{D}} = i\omega\epsilon\underline{E}$, $\underline{\dot{D}}$ cannot be $\parallel$ to z if $\underline{E}$ is transverse.

∴ Wave must be guided by $\underline{J}$ on conductors supporting $\underline{H}$ loops of wave. ∴ must be conductors for the $\underline{H}$ loops to surround, if TEM waves to propagate.

THINK: What happens in the free-space plane wave?



NO $\underline{H}$ CONFIGURATION
POSSIBLE INSIDE
HOLLOW CONDUCTOR
TRANSVERSE TO z



ALLOWED $\underline{H}$ CONFIGURATIONS ON
TWO-CONDUCTOR SYSTEMS ($\underline{H} \perp \underline{E}$)
TRANSVERSE TO z

Waveguide modes must be TE or TM, not TEM
(HOLLOW-CONDUCTOR) (Will then find $\hat{n} \neq \hat{n}_{\text{dielectric}}$).

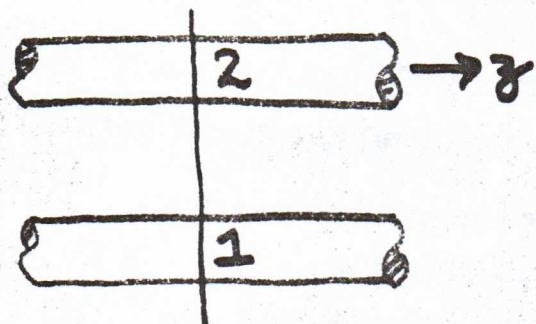
Transmission-line modes can be TEM, TE or TM
(TWO-CONDUCTOR)

TEM modes will have propagation properties set by dielectric ($\hat{n} = \hat{n}_{\text{dielectric}}$) but can be confined (coaxial line)

Transmission-line Waves (TEM modes)

Instantaneous field-configurations transverse to propagation direction are static-field configurations.

Convenient to analyse using static-field concepts:



In any (xy) plane ($z = \text{constant}$)
Potential difference
between conductors
1 and 2 at z at
any instant $V(z, t) = - \int_1^2 \underline{E} \cdot d\underline{\ell}$
between 1 and 2

Because surfaces of 1 and 2 are equipotentials at given z and t , $V(z, t)$ is independent of path of integration $1 \rightarrow 2$ provided we end in the same xy plane as we started.

Current flowing in
conductor 1 or 2
at given z, t

$$I(z, t) = \oint_{\text{around 1 or 2}} \underline{H} \cdot d\underline{\ell}$$

— also independent
of path taken in
the given xy plane

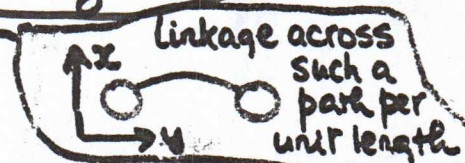
$$V(z, t) = - \int_1^2 \underline{E} \cdot d\underline{\ell} = - \int_1^2 E_x dx - \int_1^2 E_y dy \quad \text{between conductors}$$

$$\therefore \frac{\partial V}{\partial z} = - \int \frac{\partial E_x}{\partial z} dx - \int \frac{\partial E_y}{\partial z} dy$$

$$\text{i.e. } \frac{\partial V}{\partial z} = \int \frac{\partial B_y}{\partial t} dx - \int \frac{\partial B_x}{\partial t} dy$$

$$= - \frac{\partial}{\partial t} (\text{Magnetic Flux Linkage Between Wires per Unit Length})$$

using $\nabla \times \underline{E} = - \frac{\partial \underline{B}}{\partial t}$ and $E_z = 0$



$$\text{i.e. } \frac{\partial V}{\partial z} = - \frac{\partial}{\partial t} (LI) \quad \text{where } L = \text{Inductance per unit length of system}$$

$I = \text{Current at } z, t.$

$$\Rightarrow \boxed{\frac{\partial V}{\partial z} = - L \frac{\partial I}{\partial t}} \quad (67)$$

Similarly $I = \oint \mathbf{H} \cdot d\mathbf{l} = \oint H_x dx + \oint H_y dy$ around either conductor

$$\therefore \frac{\partial I}{\partial z} = \oint \left(\frac{\partial H_x}{\partial z} dx + \frac{\partial H_y}{\partial z} dy \right)$$

$$= \oint \left(+ \frac{\partial D_y}{\partial t} dx - \frac{\partial D_x}{\partial t} dy \right)$$

$$= - \frac{\partial}{\partial t} \left(\text{Electric displacement flux per unit length crossing from one conductor to other} \right)$$

$$= - \frac{\partial}{\partial t} \left(\text{Charge per unit length on the conductors} \right)$$

$$\frac{\partial I}{\partial z} = - \frac{\partial}{\partial t} (CV) \quad C = \text{Capacitance per unit length of conductors}$$

$$\therefore \boxed{\frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t}} \quad (68)$$

$$\therefore \frac{\partial^2 V}{\partial z^2} = -L \frac{\partial}{\partial t} \left(\frac{\partial I}{\partial z} \right) = LC \frac{\partial^2 V}{\partial t^2}$$

$$\frac{\partial^2 I}{\partial z^2} = -C \frac{\partial}{\partial t} \left(\frac{\partial V}{\partial z} \right) = LC \frac{\partial^2 I}{\partial t^2}$$

i.e. V and I propagate as waves in z -direction with phase velocity $1/\sqrt{LC}$

$$V = V_0 \exp i\omega(t - \sqrt{LC}z) \quad I = I_0 \exp i\omega(t - \sqrt{LC}z)$$

$$\frac{\partial V}{\partial z} = -i\omega\sqrt{LC} V \text{ also } = -L \frac{\partial I}{\partial t} = -i\omega L I$$

$$\therefore \frac{V}{I} = \frac{V_0}{I_0} = \frac{-i\omega L}{-i\omega\sqrt{LC}} = \sqrt{\frac{L}{C}} \quad \text{CHARACTERISTIC IMPEDANCE of two-wire line}$$

$$\boxed{Z_0 = \sqrt{\frac{L}{C}} \text{ specifies } V:I \text{ relation on two-wire line}}$$

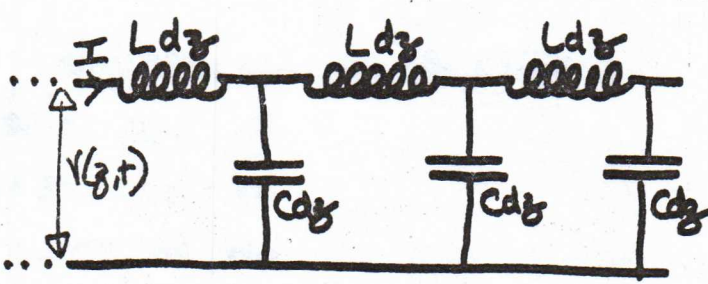
Also note that velocity of propagation of V, I must be same as velocity of propagation of E, H .

$\therefore LC \text{ must} = \mu\epsilon, \text{ whatever the line geometry}$

Distributed Circuit Analogy

Observe that Eqn. 67: $\frac{\partial V}{\partial z} = -L \frac{\partial I}{\partial t}$

Eqn. 68: $\frac{\partial I}{\partial z} = -C \frac{\partial V}{\partial t}$



are equations relating V and I in distributed circuit in diagram.
(i.e. $\partial V / \partial z = -\omega L I$ and $\partial I / \partial z = -\omega C V$)

Note: analogy valid only for TEM wave modes, in which instantaneous transverse wave fields satisfy static-field relationships from which L and C per unit length would be derived.

Lossy Lines

1) Lossy dielectric. Field analysis $i\omega\epsilon \rightarrow \sigma + i\omega\epsilon$
 $\nabla \times \underline{H} \rightarrow \nabla_t \underline{E} + \underline{\dot{D}}$

Circuit analysis
 $\frac{\partial I}{\partial z} = -(G + i\omega C) V$
 (69)

where $\sigma =$ conductivity of dielectric
 $G =$ leakage (shunt) conductance per unit length through dielectric between conductors. ($I = GV$) (leakage)

The analogy is again exact for TEM modes.

2) Finite conductivity of conducting lines.

Must now have $E_z \neq 0$, to force current through lines.

$\therefore$ no strictly TEM modes.

Cannot properly apply L, C values calculated from static-field calculations.

Field theory $\rightarrow$ Assume perfect conductors

$\rightarrow$ Derive required currents for these fields.

$\rightarrow$ Deduce required E_z distribution

$\rightarrow$ Approximate $\nabla^2 E_z$ for vector Helmholtz eqn.

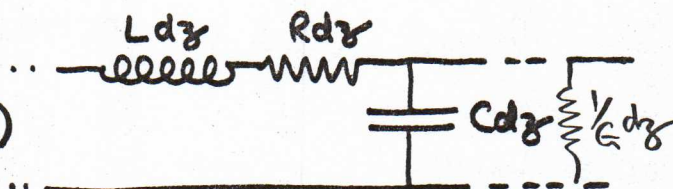
$\rightarrow$ modified propagation constants, E_x, E_y, H_x, H_y

$\rightarrow$ continue iteration.

— BUT —

In practice for reasonably efficient lines, results of field analysis are closely similar to those of distributed-circuit analysis including combined resistance R of both conductors per unit length

$$\rightarrow \boxed{\frac{\partial V}{\partial z} = -(R + i\omega L) I} \quad (70)$$



Combining Eqs. 69 and 70 for lossy dielectric and conductors

$$\rightarrow \frac{\partial^2 V}{\partial z^2} = (R + i\omega L)(G + i\omega C) V = \gamma^2 V$$

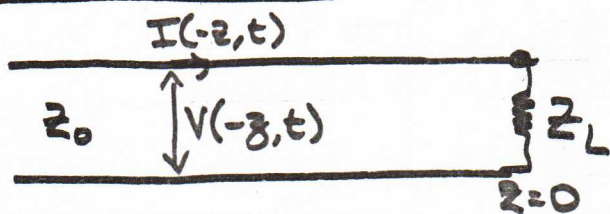
$$\frac{\partial^2 I}{\partial z^2} = (R + i\omega L)(G + i\omega C) I = \gamma^2 I$$

$$\text{Refractive index } \hat{n} = \frac{c}{i\omega} \sqrt{(R + i\omega L)(G + i\omega C)}$$

$$\text{Characteristic Impedance } Z_0 = \sqrt{\frac{R + i\omega L}{G + i\omega C}}$$

Lossy Line
Disperses,
Attenuates.

Terminated Lines



$$\frac{\partial^2 V}{\partial z^2} = \gamma^2 V \rightarrow V = V_0 e^{-\gamma z} e^{i\omega t}$$

$$\frac{\partial^2 I}{\partial z^2} = \gamma^2 I$$

$$\gamma^2 = (R + i\omega L)(G + i\omega C)$$

$\frac{V(0)}{I(0)}$ set = z_L by termination. $\frac{V(-z)}{I(-z)} = z_0$ if travelling wave.

Must $\rightarrow$ reflection from z_L unless $z_L = z_0 \rightarrow$ interference on the line

Reflection coefft (Voltage) = $\frac{z_L - z_0}{z_L + z_0} = \Gamma$ (rest is absorbed in z_L)

$$Z(-s) = \frac{V(-s)}{I(-s)} \text{ then } = z_0 \left[\frac{e^{\gamma s} + \Gamma e^{-\gamma s}}{e^{\gamma s} - \Gamma e^{-\gamma s}} \right] =$$

$$= z_0 \left[\frac{z_L \cosh \gamma s + z_0 \sinh \gamma s}{z_L \sinh \gamma s + z_0 \cosh \gamma s} \right]$$

Terminated line of variable length $\rightarrow$ variable high-frequency input impedance, used for tuning, impedance transforming in RF circuitry.

Power Calculations: $\langle P \rangle = \langle V I^* \rangle$ from circuit analogy.

Waveguide Modes - see Jackson, Ch. 8, Marion 6.7 → 6.10

Satisfy $(\nabla^2 + k_d^2) \vec{E} = 0$ subject to $\vec{n} \cdot \vec{E} = 0$ on boundaries.

For wave guided || to z , $\frac{\partial^2}{\partial z^2}$ must = $-\frac{\omega^2 \hat{n}^2}{c^2} = -k_0^2 \hat{n}^2$

Rectangular Co-ords:

Write $\vec{E}$ or $\vec{H}$ as $X(x)Y(y)Z(z)e^{i\omega t}$

$$\text{Then } \underbrace{\frac{1}{X} \frac{d^2 X}{dx^2}}_{\text{fn. of } x \text{ only}} + \underbrace{\frac{1}{Y} \frac{d^2 Y}{dy^2}}_{\text{fn. of } y \text{ only}} + \underbrace{\frac{1}{Z} \frac{d^2 Z}{dz^2}}_{\text{fn. of } z \text{ only}} + \underbrace{k_d^2}_{\text{const}} = 0$$

$$\text{i.e. } \underbrace{-A^2}_{\text{(minus signs for convenience)}} - \underbrace{B^2} - \underbrace{k_0^2 (\hat{n}^2 - n_d^2)}_{\text{must}} = 0$$

Waveguide modes are eigenfunctions of $\frac{d^2 X}{dx^2} = -A^2 X$ i.e. $X = C_1 \sin Ax + C_2 \cos Ax$
 $\frac{d^2 Y}{dy^2} = -B^2 Y$ given the bdy. Conds.
 $Y = C_3 \sin By + C_4 \cos By$

Vector Helmholtz Eqn for $\vec{E}$ $\hat{k}^2 = k_d^2 - A^2 - B^2$
→ modified propog. depends on eigenvalues A, B .

This gives DISPERSION → $\hat{n}^2 = n_d^2 - (A^2 + B^2)/k^2$

$\hat{n}$ must be $\hat{n}(\omega)$ because $\hat{c}^2, A^2, B^2$ constants

General characteristics of waveguide modes:

→ discrete modes characterised by A, B eigenvalues

dispersion (broadband distortion) increases with mode numbers.

$\hat{n} < n_d$ for real mode numbers → phase velocities $> c$.

Now return to $\nabla \times \underline{E} = -\underline{\dot{B}} = -\mu \underline{\dot{H}}$ ($\mu = \mu_{\text{dielectric}}$)

$$\rightarrow -i\omega\mu H_x = \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = \frac{\partial E_z}{\partial y} + i\omega\frac{\hat{n}}{c} E_y \quad (73)$$

$$-i\omega\mu H_y = \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -\frac{i\omega\hat{n}}{c} E_x - \frac{\partial E_z}{\partial x} \quad (74)$$

$$-i\omega\mu H_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \quad (75)$$

Similarly from $\nabla \times \underline{H} = \underline{\dot{D}} = i\omega\epsilon \underline{E}$ ($\epsilon = \epsilon_{\text{dielectric}}$)

$$i\omega\epsilon E_x = \frac{\partial H_z}{\partial y} + i\omega\frac{\hat{n}}{c} H_y \quad (76)$$

$$i\omega\epsilon E_y = -i\omega\frac{\hat{n}}{c} H_x - \frac{\partial H_z}{\partial x} \quad (77)$$

$$i\omega\epsilon E_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \quad (78)$$

Equations 73-78 can be rearranged to the form:-

$$\left. \begin{aligned} (\epsilon\mu - \frac{\hat{n}^2}{c^2}) E_x &= -\frac{i}{\omega} \left(\frac{\hat{n}}{c} \frac{\partial E_z}{\partial x} + \mu \frac{\partial H_z}{\partial y} \right) \\ (\epsilon\mu - \frac{\hat{n}^2}{c^2}) E_y &= -\frac{i}{\omega} \left(\frac{\hat{n}}{c} \frac{\partial E_z}{\partial y} - \mu \frac{\partial H_z}{\partial x} \right) \\ (\epsilon\mu - \frac{\hat{n}^2}{c^2}) H_x &= -\frac{i}{\omega} \left(\frac{\hat{n}}{c} \frac{\partial H_z}{\partial x} - \epsilon \frac{\partial E_z}{\partial y} \right) \\ (\epsilon\mu - \frac{\hat{n}^2}{c^2}) H_y &= -\frac{i}{\omega} \left(\frac{\hat{n}}{c} \frac{\partial H_z}{\partial y} + \epsilon \frac{\partial E_z}{\partial x} \right) \end{aligned} \right\} \quad (79)$$

i.e. the guided-wave modes are COMPLETELY SPECIFIED by specifying H_z and E_z as functions of x, y .

TM mode $H_z = 0$, $E_z \neq 0$ must satisfy boundary conditions

TE mode $E_z = 0$, $H_z \neq 0$ must satisfy boundary conditions

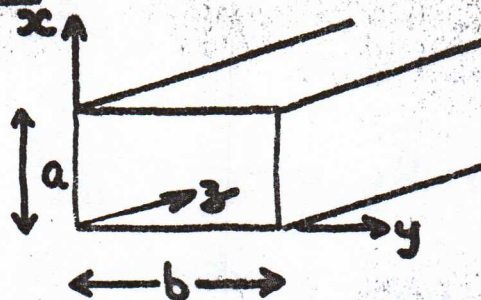
Observe that Eqs 79 $\rightarrow$ fields vanish in TEM ($E_z = H_z = 0$)

UNLESS $\epsilon\mu = \hat{n}^2/c^2$ (as we knew already).

Propagation in Rectangular Waveguides

1. TM modes. $E_z \neq 0$ generally, but:

Boundary conditions require
 $E_z \rightarrow 0$ at $x=0, x=a, y=0, y=b$



Referring back to notes, p. 114

$$E_z(x, y, z, t) \text{ must} = (C_1 \sin Ax + C_2 \cos Ax)(C_3 \sin By + C_4 \cos By) e^{i\omega(t - \frac{\hat{n}z}{c})}$$

Boundary condition at $x=0 \rightarrow C_2 \text{ must} = 0$

$$x=a \rightarrow A \text{ must} = \frac{l\pi}{a} \quad l=1, 2, 3, 4, \text{etc}$$

$$y=0 \rightarrow C_4 \text{ must} = 0$$

$$y=b \rightarrow B \text{ must} = \frac{m\pi}{b} \quad m=1, 2, 3, 4 \text{ etc}$$

$$\text{i.e. } E_z = C_1 C_3 \sin\left(\frac{l\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) e^{i\omega(t - \hat{n}z/c)}$$

$$H_z = 0$$

Notation -
 Paris/Hurd
 use m, n
 for mode
 numbers

Other field components (E_x, E_y, H_x, H_y) then follow via Eqn 79

TM wave can be completely characterised by amplitude of E_z ($= C_1 C_3$ above) and by mode numbers l and m .

Refractive index of mode is specified by mode numbers:

$$\text{Recall } \frac{1}{x} \frac{d^2 x}{dx^2} + \frac{1}{y} \frac{d^2 y}{dy^2} - \frac{\omega^2 \hat{n}^2}{c^2} + \omega^2 \mu \epsilon = 0 \quad (\text{p. 114})$$

$$\text{i.e. } \hat{n}^2 = \frac{c^2}{\omega^2} [\omega^2 \mu \epsilon - A^2 - B^2] = n_d^2 - \left(\frac{\lambda_0}{2\pi}\right)^2 \left[\frac{l^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2}\right]$$

$$\text{so: } \hat{n}^2 = n_d^2 - \frac{\lambda_0^2}{2} \left[\frac{l^2}{a^2} + \frac{m^2}{b^2}\right] \quad (80)$$

n_d = refractive index of dielectric in guide
 λ_0 = free-space wavelength

Note i) Dispersion $\hat{n}$ now depends on λ_0

ii) Cutoff Wavelength

$\hat{n}^2$ is $\oplus$ (propagation) only if

$$\lambda_0 < \frac{2 n_d}{\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}} \quad (81)$$

Note that to propagate a wave of free-space wavelength λ_0 in a waveguide, at least one dimension of the waveguide must be greater than the half-wavelength in the dielectric.

i.e. must have $a > \lambda_0/2nd$ to propagate TE_{10} mode.

All other modes (TE or TM) need larger guide dimensions.

— i.e. waveguides become impractical below $\lambda_0 \sim 30\text{cm}$ (1GHz)

Field configurations in TE_{10} mode:

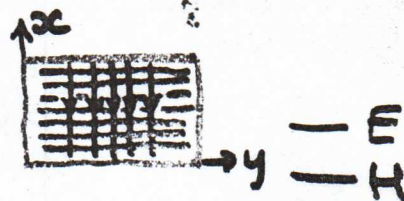
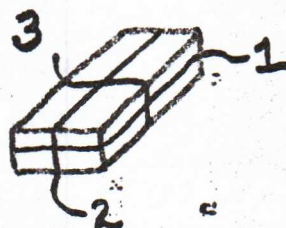
- coming
- x going



1 yz midplane



2 xz midplane



3 xy plane

Dispersion in waveguides

$$\text{Group velocity} = \frac{\omega^2 \mu \epsilon - (l\pi/a)^2 - (m\pi/b)^2}{\omega \mu \epsilon}$$

depends on l, m

Waveguide is intrinsically dispersive device — airfilled guide
has $\tilde{n}^2 = 1 - \frac{c^2[(l\pi/a)^2 + (m\pi/b)^2]}{\omega^2}$

→ behaves like IONISED GAS with $\boxed{\omega_p^2 = c^2 \left[\left(\frac{l\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2 \right]}$

Minimise distortion by arranging to propagate ONLY ONE mode, i.e. only one (l, m) pair. CAN ENSURE ONLY TE_{10} MODE PROPAGATES BY APPROPRIATE CHOICE OF GUIDE DIMENSIONS.

1. ideal materials and losses in waveguides

1) Lossy dielectric

Simply include $n_d = n_r - i n_i$ in all expressions involving refractive index of dielectric

$$\text{i.e. } n_d^2 = \mu_r \epsilon_r - \frac{i \sigma \mu_r}{\omega \epsilon_0} \quad (\text{from } (\nabla^2 + \mu_r \epsilon_r \omega^2 - i \omega \sigma \mu_r) \mathbf{E} = 0)$$

2) Imperfect walls.

Provided wall conductivity is high, field config. in guide will be very similar to that in case with $\sigma \rightarrow \infty$.

Inside real metal

$$\underline{\mathbf{E}} = \underline{\mathbf{E}}_0 e^{-y/\delta}$$

$$\underline{\mathbf{J}} = \sigma \underline{\mathbf{E}}$$

So ^{average} total power dissipated under unit surface area:

$$\bar{P} = \frac{\sigma E_0^2}{2} \int_0^\infty e^{-2y/\delta} dy$$

$$= \frac{\sigma E_0^2 \delta}{4} = \frac{J_0^2 \delta}{4\sigma}$$

The ^{amplitude of $\mathbf{H}_t$} total current flowing under unit area $I_0 = \int_0^\infty J_0 e^{-y/\delta} dy = J_0 \delta$

$$\text{i.e. } \bar{P} = \frac{I_0^2}{4\sigma\delta}$$

In limiting case $\delta \rightarrow 0$, $I_0 \rightarrow \Delta H_{t0} \rightarrow H_{t0}$ H_t = tangential $\underline{\mathbf{H}}$

$$\text{i.e. } \bar{P} \Rightarrow \frac{H_{t0}^2}{4\sigma\delta}$$

i.e. Calculate H_t as if conductors were perfect

$\rightarrow$ first-order description of losses under unit area. Integrate $\rightarrow$ total.

But boundary condition $\underline{n} \times \underline{H}_t = \underline{I}$ fixes $|I|_{(WALL)} = |H_t|_{(CAVITY)}$ at boundary

$\therefore$ Power dissipated/unit wall area = $\frac{H_t^2}{2\sigma\delta} = \frac{1}{2} H_t^2 R_s$ — surface resistance per square of the conductor
(dropping i , which is phase factor)

$\therefore$ Can approximate losses in real walls by taking H_t from $\sigma = \infty$ approximation for fields and R_s from calculation for real (not quite perfect) conducting wall.

Then to find losses in given mode must use distribution of H_t over the walls as calculated for that mode. (Messy sums)

Then writing attenuation of all wave fields as $e^{-\alpha z}$

$$\alpha = \frac{2R_s}{a\eta\sqrt{1-f_c^2/f^2}} \left\{ \left(1 + \frac{a}{b}\right) \left(\frac{f_c}{f}\right)^2 + \left[1 - \left(\frac{f_c}{f}\right)^2\right] \left[\frac{\frac{a}{b} \cdot \left(\frac{b}{a}l^2 + m^2\right)}{\frac{a^2l^2}{b^2} + m^2} \right] \right\}$$

for the TE_{lm} mode

$$\alpha = \frac{2R_s}{a\eta\sqrt{1-f_c^2/f^2}} \left[\frac{l^2(a/b)^3 + m^2}{l^2(a/b)^2 + m^2} \right]$$

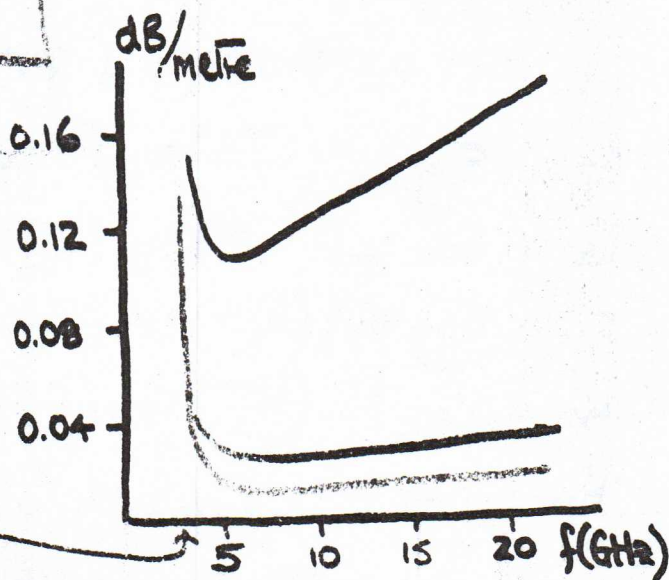
for the TM_{lm} mode

η = impedance of dielectric
 f_c = guide cutoff frequency
 f = wave frequency
 R_s = surface resistance per square of walls

Curves show attenuation of copper waveguide, $b=2$ inches, for TE_{01} mode.

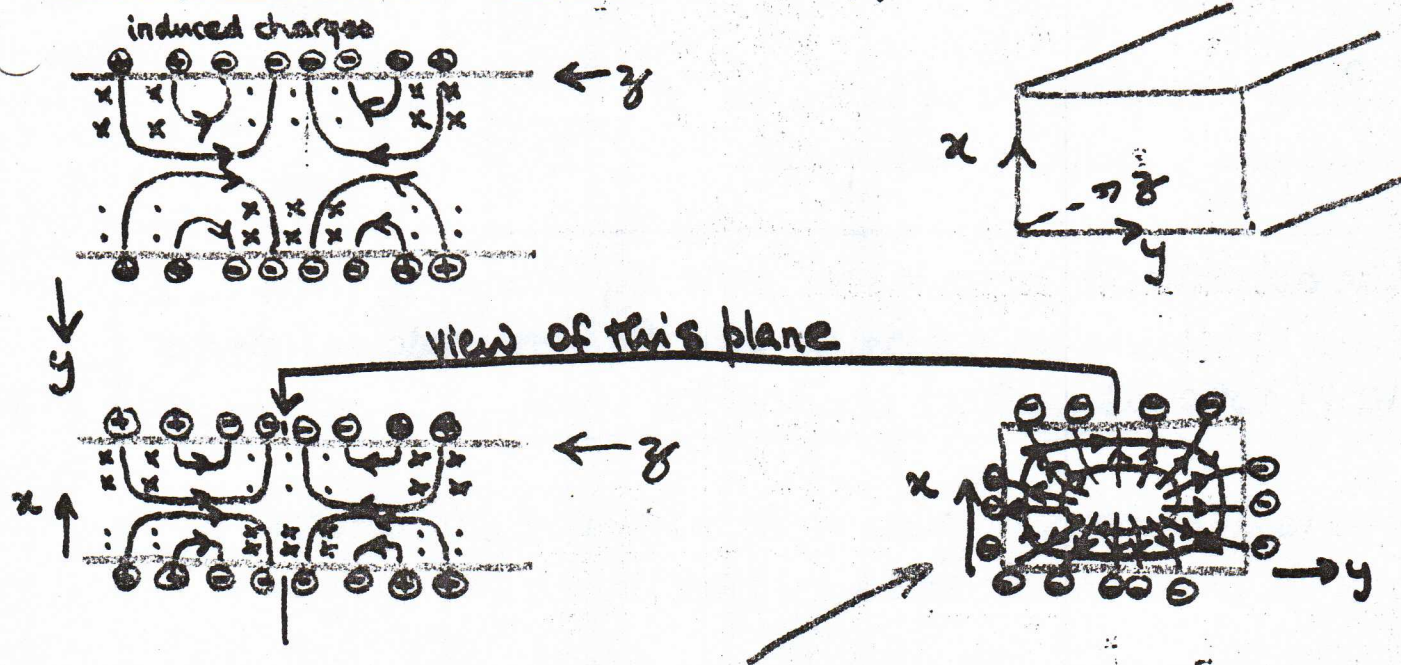
— $a = 0.2$ inches
— $a = 1$ inch
— $a = 2$ inches

CUTOFF
FREQ
~ 3GHz



Total fields in TM_{11} mode:

$\Rightarrow \vec{E}$ out
 $\Rightarrow \vec{H}$ in



Note the transverse-plane fields. Lines of $\vec{H}$ are loops in xy plane, just as in a coaxial transmission line.

But these loops close around $\vec{D}$, not $\vec{J}$ — instead of circling CONDUCTION CURRENT on central conductor, they circle the axial DISPLACEMENT CURRENT arising from $E_z(t)$.

The lines of $\vec{E}$ terminate $\perp$ to the conductors but do not radiate out from enclosed charges. Instead they bend towards the z -axis and terminate on charges further down the guide walls.

THE ENTIRE PATTERN PROPAGATES $\parallel$ to z at $c/\sqrt{\epsilon}$.

$$E_z = E_{0z} \sin\left(\frac{k\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \exp i\omega\left(t - \frac{\hat{n}z}{c}\right)$$

$$E_x = E_{0x} \cos\left(\frac{k\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \exp i\omega\left(t - \frac{\hat{n}z}{c}\right)$$

$$E_y = E_{0y} \sin\left(\frac{k\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \exp i\omega\left(t - \frac{\hat{n}z}{c}\right)$$

$$H_x = H_{0x} \sin\left(\frac{k\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \exp i\omega\left(t - \frac{\hat{n}z}{c}\right)$$

$$H_y = H_{0y} \cos\left(\frac{k\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) \exp i\omega\left(t - \frac{\hat{n}z}{c}\right)$$

$$H_z = 0$$

Sin terms
 $\rightarrow 0$ at
walls.

cos terms
 $\rightarrow 1$ at walls

b) TE modes. $E_z = 0$ everywhere requires charges on walls
 As E_x, E_y vary in time, must be currents in walls
 Hence H_z must exist at boundaries

$$H_z(x, y, z, t) = (C_1 \sin Ax + C_2 \cos Ax)(C_3 \sin By + C_4 \cos By) e^{i\omega(t - \hat{n}z/c)}$$

E-Boundary condition at $x=0 \rightarrow C_1 \text{ must} = 0$

$$x=a \rightarrow A \text{ must} = \frac{l\pi}{a} \quad l=0, 1, 2, 3, 4$$

$$y=0 \rightarrow C_3 \text{ must} = 0$$

$$y=b \rightarrow B \text{ must} = \frac{m\pi}{b} \quad m=0, 1, 2, 3, 4$$

i.e. $H_z = C_2 C_4 \cos\left(\frac{l\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) e^{i\omega(t - \hat{n}z/c)}$

$$E_z = 0$$

other field components again follow via Eqn. 79.

$$\begin{aligned} \rightarrow E_x &= E_{0x} \cos\left(\frac{l\pi x}{a}\right) \underline{\sin}\left(\frac{m\pi y}{b}\right) \exp i\omega(t - \hat{n}z/c) & \text{sin terms} \\ E_y &= E_{0y} \underline{\sin}\left(\frac{l\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \exp i\omega(t - \hat{n}z/c) & \rightarrow 0 \text{ at} \\ H_x &= H_{0x} \underline{\sin}\left(\frac{l\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \exp i\omega(t - \hat{n}z/c) & \text{the walls.} \\ H_y &= H_{0y} \cos\left(\frac{l\pi x}{a}\right) \underline{\sin}\left(\frac{m\pi y}{b}\right) \exp i\omega(t - \hat{n}z/c) \end{aligned}$$

Note that can have $l=0$, or $m=0$ in TE mode, BUT NOT BOTH
 (Both $= 0$ sends all transverse fields identically $\rightarrow 0$, leaving
 only a constant H_z along the waveguide).

As for TM modes, $\hat{n}^2 = n_d^2 - \left(\frac{\lambda_0^2}{2}\right) \left[\frac{l^2}{a^2} + \frac{m^2}{b^2} \right] \quad (80)$

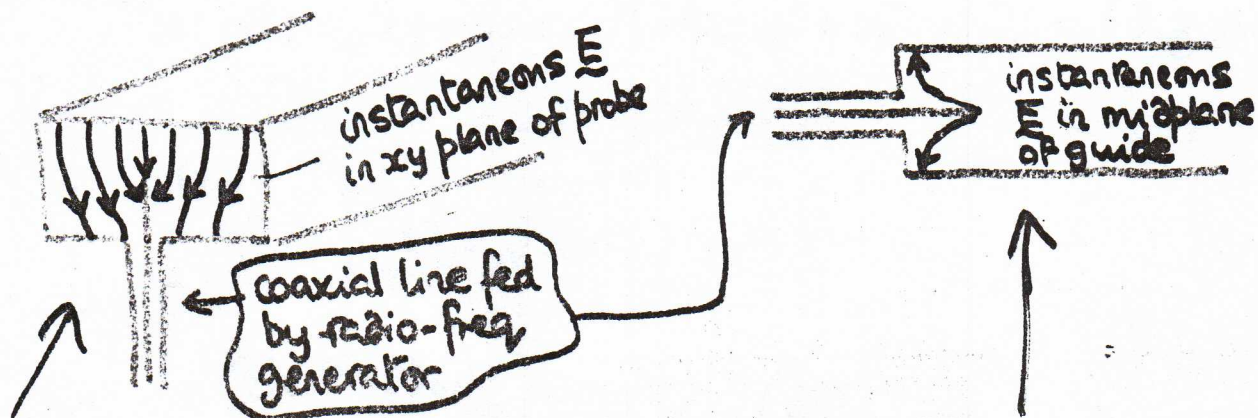
$$\lambda_{\max} = \frac{2n_d}{\left(\frac{l^2}{a^2} + \frac{m^2}{b^2}\right)^{1/2}} \rightarrow \text{maximum propagating free-space wavelength.} \quad (81)$$

Eqn. 81 $\rightarrow$ longest free-space wavelength which can propagate in rectangular waveguide whose longer dimension is a is

$$\boxed{\lambda_0 = 2n_d a} \text{ propagating in TE}_{10} \text{ mode}$$

Selective mode-launching

either introduce time-varying line charge at places in guide where E antinodes occur —

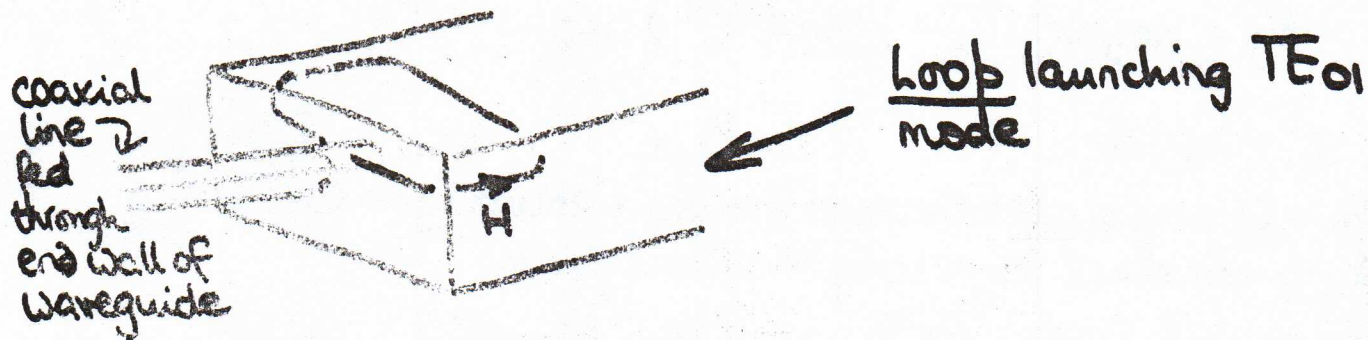


Probe launching TE_{01} mode

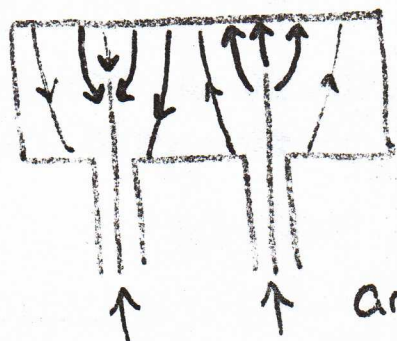
Probe launching TM_{11} mode

Exact configuration of probe in guide depends on impedance-matching needs (messy detail!)

or introduce time-varying current loop in configuration to be threaded by H of wave mode —



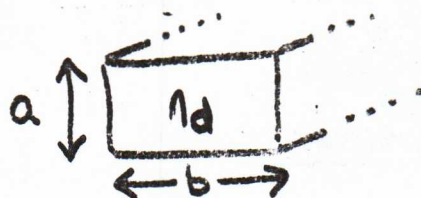
High-order modes require multiple probes introducing charges or currents varying in correct PHASE relations:-



← Antiphase probes launching TE_{02} mode

antiphase radio-frequency signals

Practical considerations (TE₀₁ mode)



1. Cutoff frequency $f_c = c/2n_d b$ } independent of short dimension of guide (a)
 wavelength $\lambda_c = 2n_d b$ } (longside)
 2. Wave propagates with linear polarisation // to short dimension of guide.
 3. Efficient launching from probe parallel to short dimension or from loop perpendicular to long dimension of guide
 4. Smaller $a \rightarrow$ greater E across guide for given power transfer, greater danger of dielectric breakdown.
 5. Power flow $\propto$ area of guide
 Wall losses $\propto$ perimeter of cross-section
 So attenuation increases as a decreases, keeping all else constant.
 6. Lowest attenuation at wave frequencies $\sim 2f_c$.
- Optimum: choose b just smaller than λ_0/n_d (to ensure no TE₀₂)
 a just smaller than $\lambda_0/2n_d$ (no TE₁₁)

Circular waveguides, see RAMO, WHINNERY VAN DUZER
 p. 429 - 437

Cylindrical Waveguides

Use cylindrical polar co-ords r, ϕ, z

$$\nabla^2 = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} \right)$$

Solve as $E = R(r)\Phi(\phi)$

TE modes $H_z = C J_n(k_c r) \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \quad k_c^2 = k_d^2 - k^2$

$$\begin{cases} E_r = i n \sqrt{\frac{\mu_d}{\epsilon_d}} \left(\frac{V}{V_c} \right) \frac{1}{k_c r} C J_n(k_c r) \begin{cases} \sin n\phi \\ -\cos n\phi \end{cases} \\ E_\phi = i \sqrt{\frac{\mu_d}{\epsilon_d}} \left(\frac{V}{V_c} \right) C J_n'(k_c r) \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \end{cases}$$

where: cutoff freq $V_c = \frac{p'_{nm}}{2\pi a n d} \quad a = \text{guide radius}$

p'_{nm} is the m 'th root of $J_n'(x) = 0$
and $(k_c)_{nm} = p'_{nm}/a$.

(boundary conditions require $J_n'(k_c a) = 0$)

TM modes Similarly $E_z = C J_n(k_c r) \begin{cases} \cos n\phi \\ \sin n\phi \end{cases}$

$$\begin{cases} H_r = -i n \frac{V}{V_c} \frac{1}{k_c r} \cdot \sqrt{\frac{\epsilon_d}{\mu_d}} C J_n(k_c r) \begin{cases} \sin n\phi \\ -\cos n\phi \end{cases} \\ H_\phi = -i \frac{V}{V_c} \cdot \sqrt{\frac{\epsilon_d}{\mu_d}} C J_n'(k_c r) \begin{cases} \cos n\phi \\ \sin n\phi \end{cases} \end{cases}$$

Boundary conditions require $J_n(k_c a) = 0$, m 'th root is p_{nm}
 $\rightarrow V_c = p_{nm}/2\pi a n d \quad (k_c)_{nm} = p_{nm}/a$

Advantages of circular guides

TE_{01} mode attenuation decreases monotonically with frequency in guide of fixed dimension

→ limiting hi- ν transmission "without losses"

TE_{01} is ~~not~~ ^{not lowest} cutoff freq. in circular guide $\left. \begin{array}{l} p'_{11} = 1.84 \\ p'_{01} = 3.83 \end{array} \right\}$

So must use mode-filtering to suppress TE_{11} launching.

TE_{01} does not require longitudinal currents (only E_ϕ exists, not E_r)
→ so can make guide out of a tight helix, or cut it transverse to the propagation direction.

This technique used by Bell in hi-freq. waveguide multiplexing.
Surround helical guide by lossy material to absorb unwanted mode fields. → no increased loss for TE_{01} , other modes severely attenuated.

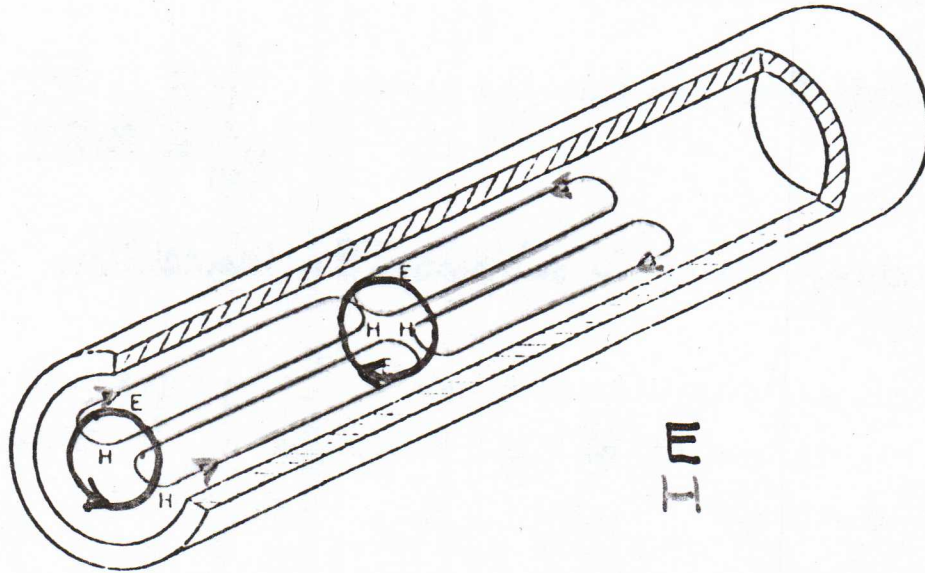


Figure II-2 The E and H field configurations for the TE_{01} mode in circular waveguide.

NO LONGITUDINAL CONDUCTIVITY NEEDED
 Waveguide wall is wire helix
 Modes other than circular electric leak out
 and are absorbed in lossy outer packing

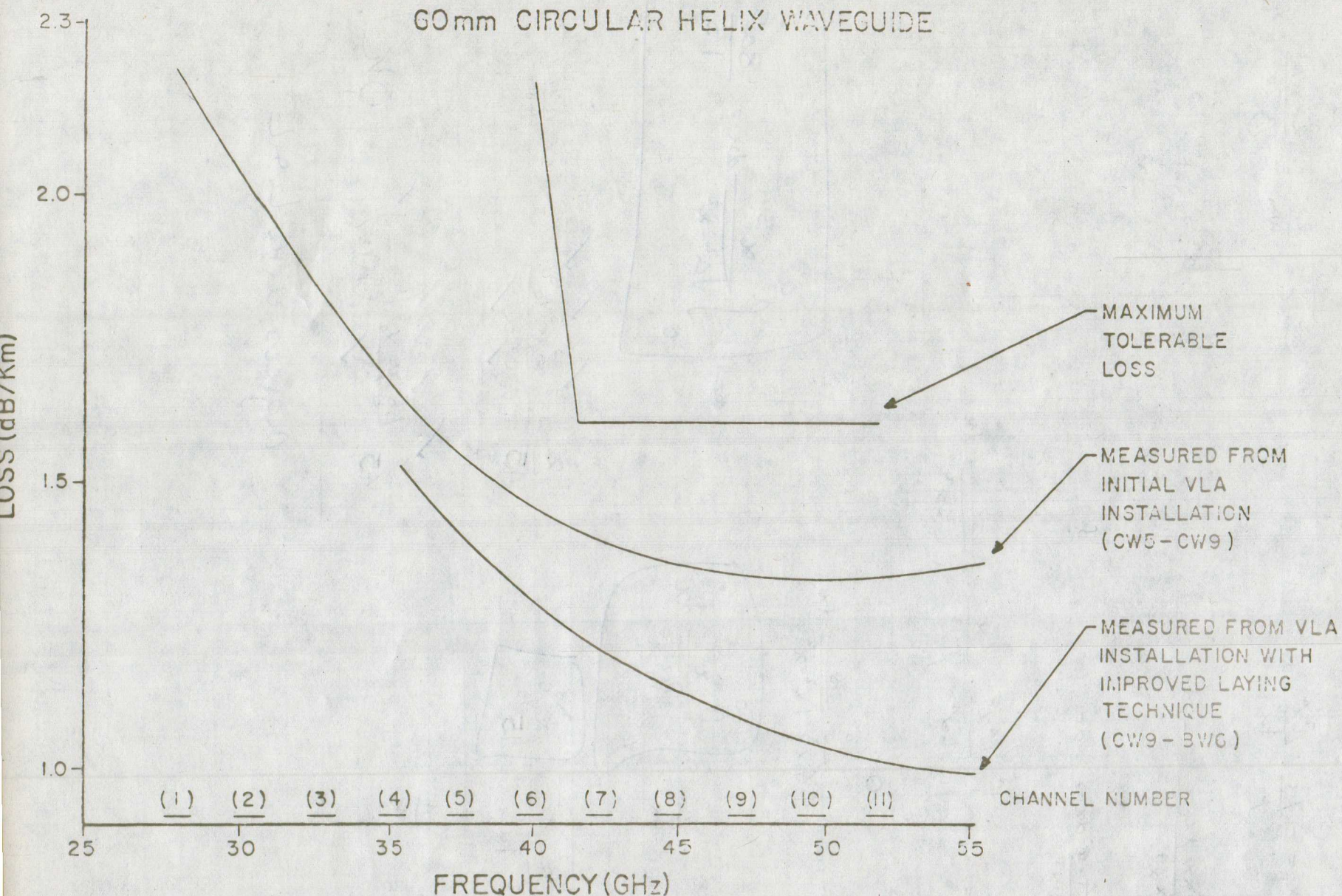


Figure II-3 Attenuation curves for the TE_{01} mode waveguide. With the initial laying technique the attenuation was found to increase during the backfilling of the trench, and a slow attenuation increase continued during subsequent months, attributable to soil settlement. The improved technique involves better compacting of the soil around the waveguide before backfilling, and subsequent attenuation increases are smaller. Both measured curves show data taken about two months after burial.