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Dear Dr. Bridle :

In a recent lecture by D.L.Jauncey here at Cornell your paper in Nature, 224, November 29, 1969, was mentioned. I formulated the problem mathematically how components superimpose and Prof. Y. Terzian urged me to send you a copy of that paper, which you will find enclosed.

Yours scincerely

A. Benz

Arnold O. Benz

Superposition of Power Spectra of Extended Radio Sources

In radio spectra of synchrotron sources one often observes that the flux $S(\nu)$ decreases according to a power law $B \cdot \nu^{-\alpha}$, where α is constant for a large range of frequencies, and one is tempted to conclude that the electron distribution in energy has to be the same throughout the source. We will show that this doesn't have to be the case.

We consider a source consisting of two components and assume that each of them has its own power law $S_i = B_i \nu^{-\alpha_i}$. For convenience we rewrite that: $S_i = C_i (\frac{\nu}{\nu_o})^{-\alpha_i}$, where ν_o is arbitrary and will be determined later. If the beam doesn't resolve the two components, they are superimposed. As A. H. Bridle[1] mentioned, it is possible to fit a new power law $S = B\,\nu^{-\alpha} = C(\frac{\nu}{\nu_o})^{-\alpha}$ to the resulting spectrum in certain cases. We will prove this in general and give an estimate of the accuracy.

$$S_1 + S_2 = C_1(\frac{\nu}{\nu_o})^{-\alpha_1} + C_2(\frac{\nu}{\nu_o})^{-\alpha_2} = C_1 e^{-\alpha_1 \ln \nu/\nu_o}$$

$$+ C_2 e^{-\alpha_2 \ln \nu/\nu_o} = C_1(1 - \alpha_1 \ln \nu/\nu_o + \frac{\alpha_1^2 \ln^2 \nu/\nu_o}{2!} + \ldots)$$

$$+ C_2(1 - \alpha_2 \ln \nu/\nu_o + \frac{\alpha_2^2 \ln^2 \nu/\nu_o}{2!} + \ldots)$$

We want to fit a function $S(\nu)$ with

$$S = C\left(\frac{\nu}{\nu_o}\right)^{-\alpha} = C\,e^{-\alpha \ln \nu/\nu_o} = C\left(1 - \alpha \ln \nu/\nu_o + \frac{\alpha^2 \ln^2 \nu/\nu_o}{2!} - + \ldots\right)$$

So in the first order (what might be different from the best fit under certain conditions as we will see later):

$$C = C_1 + C_2 \qquad \alpha = 1/2(\alpha_1 + \alpha_2)$$

and the error amounts to:

$$S_1 + S_2 - S = a_1 \ln \nu/\nu_o + a_2 \ln^2 \nu/\nu_o + \ldots$$

where:

$$a_1 = \tfrac{1}{2} C_1\alpha_1 + \tfrac{1}{2} C_1\alpha_2 + C_2\alpha_1 + \tfrac{1}{2} C_2\alpha_2 - C_1\alpha_1 - C_2\alpha_2$$

$$= \tfrac{1}{2}(C_2 - C_1)(\alpha_1 - \alpha_2)$$

$$a_2 = \tfrac{1}{2!}[C_1\alpha_1^2 + C_2\alpha_2^2 - \tfrac{1}{4}C_1\alpha_1^2 - \tfrac{1}{2}C_1\alpha_1\alpha_2 - \tfrac{1}{4}C_1\alpha_2^2 - \tfrac{1}{4}C_2\alpha_1^2$$

$$- \tfrac{1}{2}C_2\alpha_1\alpha_2 - \tfrac{1}{2}C_2\alpha_1^2] = \tfrac{1}{2}(\alpha_1 - \alpha_2)[(\alpha_1 + \alpha_2)(C_1 - C_2)$$

$$- \tfrac{1}{2}(C_1 - C_2)(\alpha_1 - \alpha_2) + C_1(\alpha_1 - \alpha_2)]$$

$$\vdots$$

$$a_n = \frac{1}{n!}[C_1\alpha_1^n + C_2\alpha_2^n - \frac{1}{2n}(\alpha_1 + \alpha_2)^n(C_1 + C_2)]$$

$$= \frac{(\alpha_1 + \alpha_2)}{2n} a_{n-1} + \frac{(\alpha_1 - \alpha_2)}{2n!}[C_1\alpha_1^{n-1} - C_2\alpha_2^{n-1}]$$

All coefficients are products of $(\alpha_1-\alpha_2)$ or (C_1-C_2). We consider the approximation within a certain range only. We assume that the point of intersection of the two power-spectra is within that range, because this is the only case where we expect to resolve their spectra anyway. We therefore put ν_o equal to the frequency of intersection. $S_1 = S_2$ $C_1 = C_2$. a_1 vanishes, and if the first correction term is a good estimate of the error, i.e. if we are not to far away from ν_o and/or $|\alpha_1-\alpha_2|$ is small, the ratio of the error to the actual value is:

$$\frac{S_1+S_2-S}{S_1+S_2} \cong \frac{S_1+S_2-S}{S} = \frac{1}{2}\frac{C_1}{C_1+C_2}(\alpha_1-\alpha_2)^2 \frac{\ln^2 \nu/\nu_o}{(\nu/\nu_o)^{-\alpha}}$$

$$= \frac{1}{2}\frac{(\alpha_1-\alpha_2)^2}{\alpha_1+\alpha_2}\frac{(\ln(\nu/\nu_o)^{-\alpha})^2}{(\nu/\nu_o)^{-\alpha}}$$

$$= \frac{1}{2}\frac{(\alpha_1-\alpha_2)^2}{\alpha_1+\alpha_2} F(\nu/\nu_o)$$

where $$F(\nu/\nu_o) = \frac{[\ln(\nu/\nu_o)^{-\alpha}]^2}{(\nu/\nu_o)^{-\alpha}}$$

We note that the error is positive in the first order and gets large at higher frequencies. It depends strongly upon the difference of the two exponents and the frequency of intersection.

For $\nu_o = 500$ MHz: F(50 MHz, 500) = 0.324
F(5000 MHz, 500) = 8.0

$\nu_o = 2500$ MHz: F(50 MHz, 2500) = 0.176
F(5000 MHz, 7500) = 0.79

At a high ν_o the two components cannot be resolved in the spectrum. This fit of the average exponent is best near the frequency of intersection. For a large interval the least square fit might be slightly different and the calculated error is then an upper limit.

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[1]A. H. Bridle, Nature, <u>224</u>, 889, 1969.