RADIO ASTRONOMY

by

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These notes form the basis of a course of lectures given in the second term of the academic year 1957-8 and the first term of the academic year 1958-9 at the California Institute of Technology. They contain more material than was used in the course. In particular, the material in Section 3.3 was not discussed, and that in Chapter 5 was discussed only in a descriptive manner. It was included on account of its relative inaccessibility.

The material here is regarded as basic to radio astronomy. Not included is the substance of some six additional lectures given by J. G. Bolton, T. A. Matthews, and J. A. Roberts on recent work in galactic, 21-cm, and solar radio astronomy.

UNITS IN ELECTROMAGNETIC THEORY

The electromagnetic equations in these lecture notes are expressed in terms of a rationalized system of units and the free-space permeability and permittivity μ_v and ϵ_v. The latter quantities are related by the formula $\mu_v \epsilon_v = 1/c^2$, where c is the free-space velocity of electromagnetic radiation. Having retained both μ_v and ϵ_v explicity in the equations, one is enabled to convert these into any system of units in current use by following the scheme below.

Rationalized Electromagnetic Units

cm-gm-sec	m-kg-sec
$\mu_v = 4\pi$ (dimensionless)	$\mu_v = 4\pi \times 10^{-7}$ (dimensionless) or henry m^{-1}
$\epsilon_v = 1/4\pi c^2\ cm^{-2} sec^2$	$\epsilon_v = 10^7/4\pi c^2\ m^{-2} sec^2$ or farad m^{-1}
where $c = 2.998 \times 10^{10}$ cm sec^{-1}	where $c = 2.998 \times 10^8$ m sec^{-1}

Rationalized Electrostatic Units - cm-gm-sec

$\epsilon_v = 4\pi$ (dimensionless)

$\mu_v = 1/4\pi c^2\ cm^{-2}\ sec^2$

where $c = 2.998 \times 10^{10}$ cm sec^{-1}

Unrationalized Units

The equations may be expressed in unrationalized form by first converting from $\underset{\approx}{H}$ and $\underset{\approx}{D}$ to $\underset{\approx}{B}$ and $\underset{\approx}{E}$, and then replacing μ_v and ϵ_v by $4\pi\mu_v$ and $\epsilon_v/4\pi$ wherever they occur; except that the constitutive relations

$$\underset{\approx}{H} = \underset{\approx}{B}/\mu_v, \quad \underset{\approx}{D} = \epsilon_v \underset{\approx}{E},$$

are not altered. The values of μ_v and ϵ_v in the above tables have then to be respectively divided and multiplied by 4π. It is evident that, in both rationalized and unrationalized systems, the values of the units of the electromagnetic quantities are the same except for $\underset{\sim}{H}$ and $\underset{\sim}{D}$ where they differ by a factor 4π.

Gaussian Units

This unrationalized mixed electromagnetic and electrostatic system is still favored by workers in atomic physics. To express the equations in these units they must again be expressed in terms of $\underset{\sim}{B}$ and $\underset{\sim}{E}$; then μ_v is assigned the value 4π and ϵ_v the value $1/4\pi$, except that now $\underset{\sim}{H} = \underset{\sim}{B}$ and $\underset{\sim}{D} = \underset{\sim}{E}$. At the same time $\underset{\sim}{J}$, $\underset{\sim}{j}$, and the conductivity σ must be replaced by $\underset{\sim}{J}/c$, $\underset{\sim}{j}/c$, and σ/c; and, in differential equations relating the field vectors and source functions, the operation $\partial/\partial t$ must be replaced by $(\partial/\partial t)/c$. Thus, the magnetic field is expressed in electromagnetic units and the electric field, charge, and current, in electrostatic units.

CORRIGENDA

p. 15, footnote: The example refers to the spectrum of a single pulse occurring at time $t = 0$. Noise may be represented by a superposition of impulses of different strengths occurring at random intervals.

p. 23: In the last sentence of the second paragraph replace "strength" by "angular width".

p. 36: The exponent in Eq.(14) should be

$$-i\omega\,(t - q\,\underset{\sim}{n}\cdot\underset{\sim}{r}/c)\ .$$

p. 38: In the sixth line from the bottom the inequality should be reversed.

p. 45: The elimination referred to after Eq.(15) is achieved simply by substituting from the result

$$\underset{\approx}{K}\cdot\underset{\sim}{E} = \underset{\sim}{E} - \underset{\sim}{j}/i\omega\,\epsilon_v$$

in (13), and using (3).

p. 48: In the results from Eq.(17) "0" and "∞" should be interchanged.

Figure 2 (opp. p. 64): The trajectories bounding $d\Omega_1$ should pass through the boundary of dS_2 if continued from dS_0 into medium 2; likewise, those bounding $d\Omega_2$ should pass through the boundary of dS_1 if continued back into medium 1. Thus, in the corrected figure, dS_1 and dS_2 are much larger than drawn here.

p. 72: In Eq.(1) the first " v " should be " $\mathring{v}$ ".

p. 74: In Eq.(9) the last bracket should be "$(1 + \overline{\upsilon_{01}}^2)$".

p. 97: In the last line of the first paragraph "V_g" should be "$\|v_g\|$".

p. 98: In the last term in the bottom line "f" should be "f_0".

Figure 2 (opp. p. 124): The great-circle arc between $\underset{\sim}{k}$ and $\underset{\sim}{v}$ should be designated by "α".

p. 123: The formula preceding Eq.(3) should be

$$R = r(1 - \underset{\sim}{r}_1 \cdot \underset{\sim}{r}/r^2) .$$

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CHAPTER 1

The Observation of Radio-Frequency Radiation

1.1 Introduction

Practically all our knowledge of the Universe beyond this planet is gained from our observations of the radiation that penetrates the Earth's atmosphere. This atmosphere is impenetrable to the greater part of the forty octaves of the electromagnetic spectrum wherein it is possible to take measurements. There are two windows in this blanket - the first admits about one octave in the visual region of the spectrum, and the second some fourteen octaves in the radio band between wavelengths of about 1 cm and 10 meters.

Up till the last war, practically all observations made were of radiation passing through the first window. These probably began when men first looked through their eyes. The systematic observation of radiation passing through the second window dates from the last war, by which time man had devised radio equipment sufficiently sensitive to measure the radiation that reaches us. It is interesting to speculate in passing on the possible course of the history of astronomy, indeed of this planet, if man had evolved eyes that were sensitive to radiation passing through the second window rather than the first. In this course we shall be concerned with the radio-frequency radiation that reaches us through the second window. We shall see that, in general, such observations [illegible] to relatively diffuse bodies, e.g. the atmospheres of stars and clouds of interstellar gas, wherein radio-frequency [illegible] [illegible] propagated without serious inhibition.

R A D I O   A S T R O N O M Y

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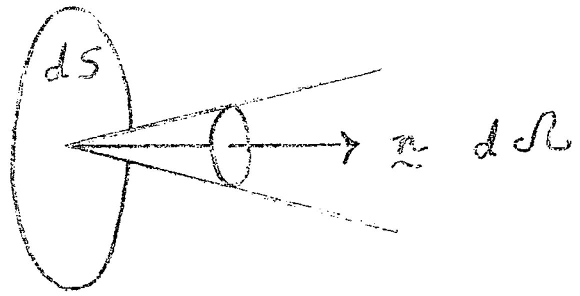

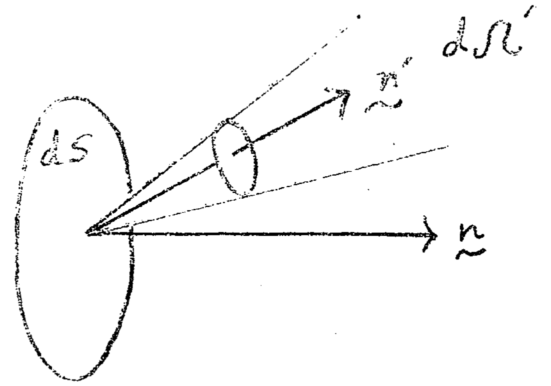

Figure 1. The intensity of radiation

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1.2 Intensity, Brightness and Flux Density of Radiation

Consider the total radiation, i.e. the radiation over the whole spectrum, at some point in space at some instant. Its total intensity $I(\underset{\sim}{n})$ in the direction $\underset{\sim}{n}$ is such that radiant energy of amount

$$I dS d\Omega dt$$

passes normally through an element of area dS into the solid angle $d\Omega$ in time dt.

It follows that the radiant power flowing into the solid angle $d\Omega' \equiv d\Omega(\underset{\sim}{n}')$ about the direction $\underset{\sim}{n}'$ is

$$I(\underset{\sim}{n}') \quad \underset{\sim}{n}'\cdot\underset{\sim}{n}\, dS\, d\Omega(\underset{\sim}{n}') ,$$

which we may write

$$I' \; \underset{\sim}{n}'\cdot\underset{\sim}{n}\, dS\, d\Omega' \quad .$$

If $I(\underset{\sim}{n})$ = constant at a point, the radiation field there is isotropic.

[M.k.s. unit: watt m^{-2} steradian^{-1}]

Monochromatic intensity. Radio receivers and transmitters in general operate efficiently only over a narrow band of frequencies Δf about some central frequency f; and in addition accept only one kind of polarization of this "monochromatic" radiation. The monochromatic intensity I_f is such that the total radiation I is given by

$$I = \int_0^\infty I_f \, df \; . \qquad (1)$$

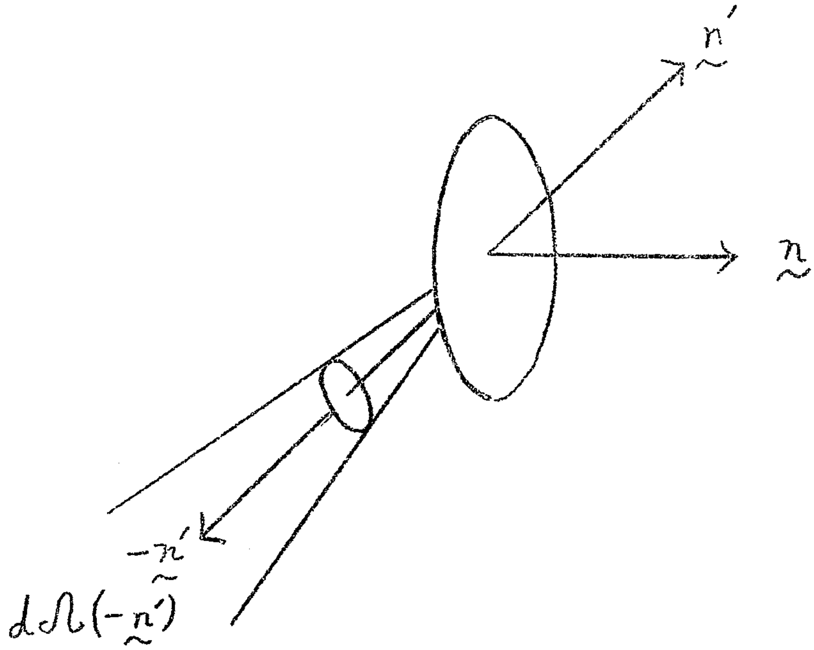

Figure 2. The brightness of radiation

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Then

$$I_f' \, df \; \underset{\sim}{n}' \cdot \underset{\sim}{n} \; dS \; d\Omega'$$

is the radiant power in the range $(f, f+df)$, flowing through the surface $\underset{\sim}{n}dS$ into the solid angle $\underset{\sim}{n}' \, d\Omega'$, at any instant. Strictly speaking, since I_f is obtained by Fourier analysis of a function of the time, involving integration from $t = -\infty$ to $t = +\infty$, it must itself be independent of the time. In practice, however, the receiving equipment has an overall time constant τ, so that what is recorded is effectively the detected signal smoothed by successive averaging over time intervals of duration τ. Clearly, the time constant must not be so large that significant variations in intensity are smoothed away. We shall see in Sec. 1.5 that a lower limit on the value of τ is set by the requirements of receiver sensitivity.

Brightness. This quantity is defined in the same way as intensity, save that the associated directions are reckoned as those from which the radiation is coming rather than those in which it is going. Thus

$$b_f(\underset{\sim}{n}') df \; \underset{\sim}{n}' \cdot \underset{\sim}{n} \; dS \; d\Omega'(\underset{\sim}{n}')$$

i.e.

$$b_f' df \; \underset{\sim}{n}' \cdot \underset{\sim}{n} \; dS d\Omega'$$

is the radiant power in the range $(f, f+df)$ incident on $\underset{\sim}{n}dS$ from the solid angle $\underset{\sim}{n}' d\Omega'$. It follows that

$$b_f(\underset{\sim}{n}') d\Omega(\underset{\sim}{n}') = I_f(-\underset{\sim}{n}') d\Omega(-\underset{\sim}{n}'). \qquad (2)$$

The concept of brightness is appropriate to observations at great

distances from the sources of the radiation.
[M.k.s. unit: watt $m^{-2}(c/s)^{-1}$ steradian^{-1}]

Flux Density - Specification of a Discrete Source. Suppose the radiation from a source is contained within a solid angle $\Delta\Omega$, then the flux density of f-radiation from the source is defined as

$$S_f = \int_{\Delta\Omega} b_f{}' \, d\Omega' = \int_{\Delta\Omega} I_f(-\underset{\sim}{n}') \, d\Omega(-\underset{\sim}{n}'). \qquad (3)$$

In this specification it is assumed that $\Delta\Omega$ is not so large that $\underset{\sim}{n}'$ does not remain sensibly constant over the source. Thus

$$S_f \, df \, dS$$

is the radiant power from the source in the band (f, f+df) incident normally on a surface dS.
[M.k.s. unit: watt $m^{-2}(c/s)^{-1}$]

Temperature Specification of Radiation

Brightness Temperature. For black-body radiation, the monochromatic intensity, or brightness, everywhere in an enclosure and in all directions is given by the Planck formula, which, for radio frequencies is adequately represented by the Rayleigh-Jeans formula

$$B_f = 2\frac{k}{c^2}f^2T = 2\frac{k}{\lambda^2}T, \qquad (1)$$

where T is the temperature of the enclosure, f the frequency, k Boltzmann's constant and c the velocity of electromagnetic radiation in free space; λ is then the wavelength in free space. The factor 2 is to take account of the two polarizations into which it is possible to resolve unpolarized radiation; only one polarization is accepted by any given antenna system. The brightness temperature of the radiation from the direction $\underset{\sim}{n}$, of the polarization accepted by the antenna or aerial system, is defined as the temperature of black-body radiation which would have the same brightness $b_f(\underset{\sim}{n})$ at the aerial. Thus

$$b_f(\underset{\sim}{n}) = \frac{k}{c^2} f^2 T_b(\underset{\sim}{n}). \qquad (2)$$

[$k/c^2 = 1.54 \times 10^{-40}$ kg $(^oK)^{-1}$]

Apparent Disk Temperature. The apparent (disk) temperature T_s of a source subtending the angle $\Delta\Omega$ at the aerial is defined as the temperature of black-body radiation which would give the same flux density from the same solid angle, as the incident radiation from the source of the polarization accepted by the aerial.

Thus, by Sec. 1.2(3), and (2),

$$S_f = \frac{k}{c^2} f^2 \int_{\Delta\Omega} T_b' d\Omega' = \frac{k}{c^2} f^2 T_s \Delta\Omega , \qquad (3)$$

giving

$$T_s = \frac{1}{\Delta\Omega} \int_{\Delta\Omega} T_b' d\Omega' . \qquad (4)$$

The angle $\Delta\Omega$ subtended by the source may be arbitrarily specified e.g. for the Sun it is usual to take for $\Delta\Omega$ the angle subtended by the photospheric disk while the brightness integration is taken over the whole source region, which includes the corona. In any case, the strength of a source is determined by the product $T_s \Delta\Omega$ -- equivalent to a source of uniform brightness temperature T_s subtending an angle $\Delta\Omega$.

[Note that the figures usually quoted for flux density refer to the quantity 2 S_f as given by (3). This follows from the implicit assumption that the radiation is unpolarized so that 2 S_f is the flux density of the incident radiation.]

4. Antenna Relationships

Effective Aerial Temperature. A receiving aerial or antenna is specified by its effective area (which includes its directional characteristics) its frequency bandwidth and its polarization. Radiation of any polarization can be resolved into two components, one of the polarization of the aerial and the other of the complementary polarization. If the radiation is unpolarized, these components are of equal power so that the total power incident is twice the power measured.

The effective area $A(\underset{\sim}{n},\underset{\sim}{n}')$ is defined so that the power P within the bandwidth Δf available at the output terminals when the aerial is directed towards $\underset{\sim}{n}$ is

$$P = \Delta f \int A(\underset{\sim}{n},\underset{\sim}{n}')\; b_f\,(\underset{\sim}{n}')\; d\Omega(n'). \qquad (1)$$

The orientation of the aerial is determined by the direction at which maximum power is received from a point source. Thus

$$A(\underset{\sim}{n},\underset{\sim}{n}) = A_{max} .$$

In the formula for incident power in the definition of brightness $n'\cdot\underset{\sim}{n}dS$ has become the effective area associated with available power. In terms of the brightness temperature of the radiation accepted

$$P = \frac{k}{\lambda^2}\;\Delta f \int A\; T_b'\; d\Omega' . \qquad (2)$$

The concept of effective aerial temperature is arrived at by comparing P with the value it would take if the aerial were in an enclosure at a uniform temperature T. We should then have

$$P = \frac{kT\Delta f}{\lambda^2} \int A \; d\Omega',$$

i.e.

$$P = \frac{4\pi}{\lambda^2} \bar{A} \; kT \; \Delta f,$$

where

$$\bar{A} = \frac{1}{4\pi} \int A \; d\Omega' \tag{3}$$

is the mean effective area. Now it can be proved by thermodynamics (Burgess 1941) that in this case the power absorbed by a <u>loss-free</u> aerial is the same as that dissipated in a resistor at the same temperature,* viz.

$$P = kT \, \Delta f.$$

Thus, for a loss-free aerial, we have

$$\bar{A} = \lambda^2/4\pi \,, \tag{3'}$$

and <u>for any aerial</u> in any radiation field we may express the available power in terms of the aerial or antenna temperature T_a

$$P = kT_a \; \Delta f, \tag{4}$$

where

$$T_a = \frac{1}{\lambda^2} \int A T_b' \; d\Omega'. \tag{5}$$

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* This is a classical result, valid for $hf \ll kT$, when stimulated emission is negligible.

Directivity and Gain. The directivity of any aerial is defined as the normalized dimensionless quantity

$$D = A/\bar{A} , \tag{6}$$

which is such that

$$\bar{D} = \frac{1}{4\pi} \int D \, d\Omega' = 1 . \tag{7}$$

The relation (5) now becomes

$$T_a = \frac{\bar{A}}{\lambda^2} \int D \, T_b' \, d\Omega' . \tag{8}$$

In particular, for a loss-free aerial

$$T_a = \frac{1}{4\pi} \int D \, T_b' \, d\Omega' . \tag{8'}$$

The directional characteristics of an aerial are sometimes specified by its gain, which is defined for an aerial qua transmitter. The gain is then the ratio of the actual flux density at some field point to the hypothetical flux density at the same point if all the power delivered to the input terminals were radiated isotropically. Thus, if $S(\underset{\sim}{n},\underset{\sim}{n}')$ is the Poynting flux (radiant power per unit area) in the direction $\underset{\sim}{n}'$ at a distance r, when the aerial is directed towards $\underset{\sim}{n}$,

$$G(\underset{\sim}{n},\underset{\sim}{n}') = \frac{S(\underset{\sim}{n},\underset{\sim}{n}')}{P/4\pi r^2} . \tag{9}$$

It can be proved by thermodynamics* that G is related to the effec-

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* Consider an aerial and a small black body subtending the angle $\Delta\Omega$ at the aerial in an enclosure at a uniform temperature T. The power absorbed by the body from the aerial is equal to the incident power $G \, kT \, \Delta f \, \frac{\Delta\Omega}{4\pi}$. Again, the power incident on the aerial is $\frac{kT \, \Delta f}{\lambda^2} A \, \Delta\Omega$, by (2). Since there is equilibrium these are equal. Hence the result (10).

tive area A of the aerial when used as a receiver by the formula

$$G = \frac{4\pi A}{\lambda^2} . \tag{10}$$

Hence, for any aerial we have from (5) the relation

$$T_a = \frac{1}{4\pi} \int G \, T_b' \, d\Omega' . \tag{11}$$

The quantities G and D are evidently related by the formula

$$G = \frac{4\pi \bar{A}}{\lambda^2} D = \bar{G} D . \tag{12}$$

It follows from our definition of G that $\bar{G}$ is the fraction of the power delivered to the input of the aerial that is actually radiated. For a loss-free aerial

$$G = D, \quad \bar{G} = 1 . \tag{12'}$$

The ideal isotropic radiator is, of course, not physically realizable. The simplest radiators are the Hertzian dipole and Fitzgerald oscillator, which are simulated respectively by a short rod whose ends are in opposite phase and by a small loop. For these, neglecting losses,

$$G = \frac{\cos^2 \theta}{\frac{1}{4\pi} \int \cos^2 \theta \, d\Omega} = \frac{3}{2} \cos^2 \theta = D,$$

where $\cos \theta = \underset{\sim}{n} \cdot \underset{\sim}{n}'$. Then, by (10),

$$A = \frac{3}{8\pi} \lambda^2 \cos^2 \theta ,$$

which satisfies (3').

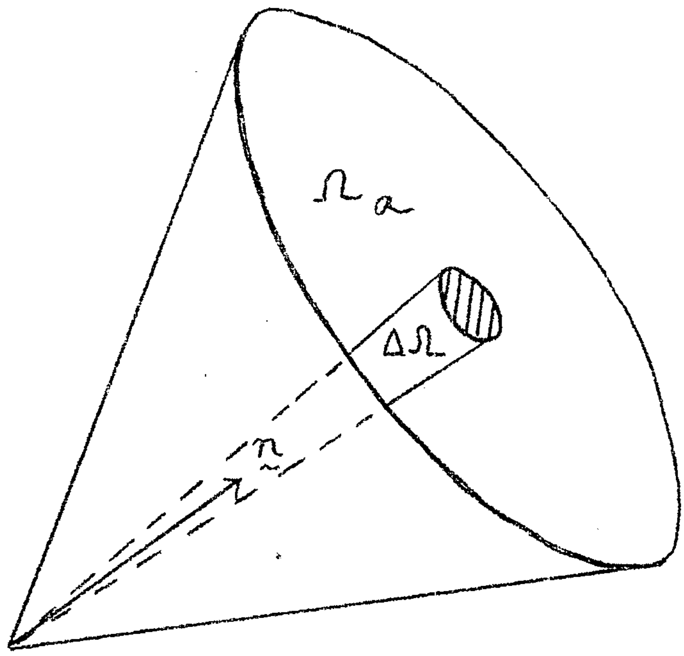

Figure 3. The effective solid angle of the aerial.

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The directional diagram, or radiation pattern, of an aerial is specified by any one of the mutually proportional quantities, A, D or G, usually in terms of the polar angles θ, ϕ of $\underset{\sim}{n}'$ relative to $\underset{\sim}{n}$, the direction of maximum response. In practice the terms "directivity" and "gain" are frequently applied to the maximum values of these functions, $D(\underset{\sim}{n},\underset{\sim}{n})$, $G(\underset{\sim}{n},\underset{\sim}{n})$. The directional diagram of an aerial, when plotted in polar coordinates, usually consists of a "main lobe" about the direction of $\underset{\sim}{n}$ and a number of subsidiary lobes. The main lobe is usually specified by the width between the "half-power" points and the gain by the maximum gain.

Aerial Temperature of a Source. Suppose that the aerial is directed towards $\underset{\sim}{n}$, and that there is a source of apparent disk temperature T_s which subtends a small angle $\Delta\Omega$ about $\underset{\sim}{n}'$. Then the aerial temperature due to the source is, by Sec. 1.3 (4), and (5), (6), (10),

$$\left.\begin{aligned} T_a &= \frac{A(\underset{\sim}{n},\underset{\sim}{n}')}{\lambda^2}\, T_s\, \Delta\Omega \\ &= \frac{\bar{\Lambda} D}{\lambda^2}\, T_s\, \Delta\Omega \\ &= \frac{G}{4\pi}\, T_s\, \Delta\Omega . \end{aligned}\right\} \tag{13}$$

Naturally one tries to direct the aerial at the source, so that $\underset{\sim}{n} = \underset{\sim}{n}'$ and $G = G_{max}$. These relations may then be expressed more physically, in terms of the "effective solid angle" Ω_a of the aerial. This is the angle such that

$$G_{max}\, \Omega_a = \int G\, d\Omega = 4\pi\bar{G}, \text{ etc.} \tag{14}$$

Then, when the aerial is directed at the source,

$$T_a = \bar{G}\, T_s \frac{\Delta\Omega}{\Omega_a}, \tag{15}$$

which is interpretable, remembering that $\bar{G}$ is the fraction of power that is not dissipated between the field and the output terminals of the aerial qua receiver. Neglecting losses,

$$T_a = T_s \frac{\Delta\Omega}{\Omega_a} . \tag{15'}$$

We remark that if (15) is used taking Ω_a as given by an integral such as (14), in which the integration is taken only over the main lobe, too small a value of $T_s\Delta\Omega$ will be inferred from a measurement of T_a. Again, if the source is so large that A, D, G vary significantly over $\Delta\Omega$, (13) must be replaced by

$$T_a = \frac{1}{\lambda^2} \int_{\Delta\Omega} A T_b' \, d\Omega', \text{ etc.}$$

Finally we give the relations between the aerial temperature T_a and flux density S_f (for the polarization accepted by the aerial). By Sec. 1.3(3), and (13), these are

$$\left.\begin{aligned} kT_a &= AS_f \\ &= \bar{A}DS_f \\ &= \frac{G\lambda^2}{4\pi} S_f . \end{aligned}\right\} \tag{16}$$

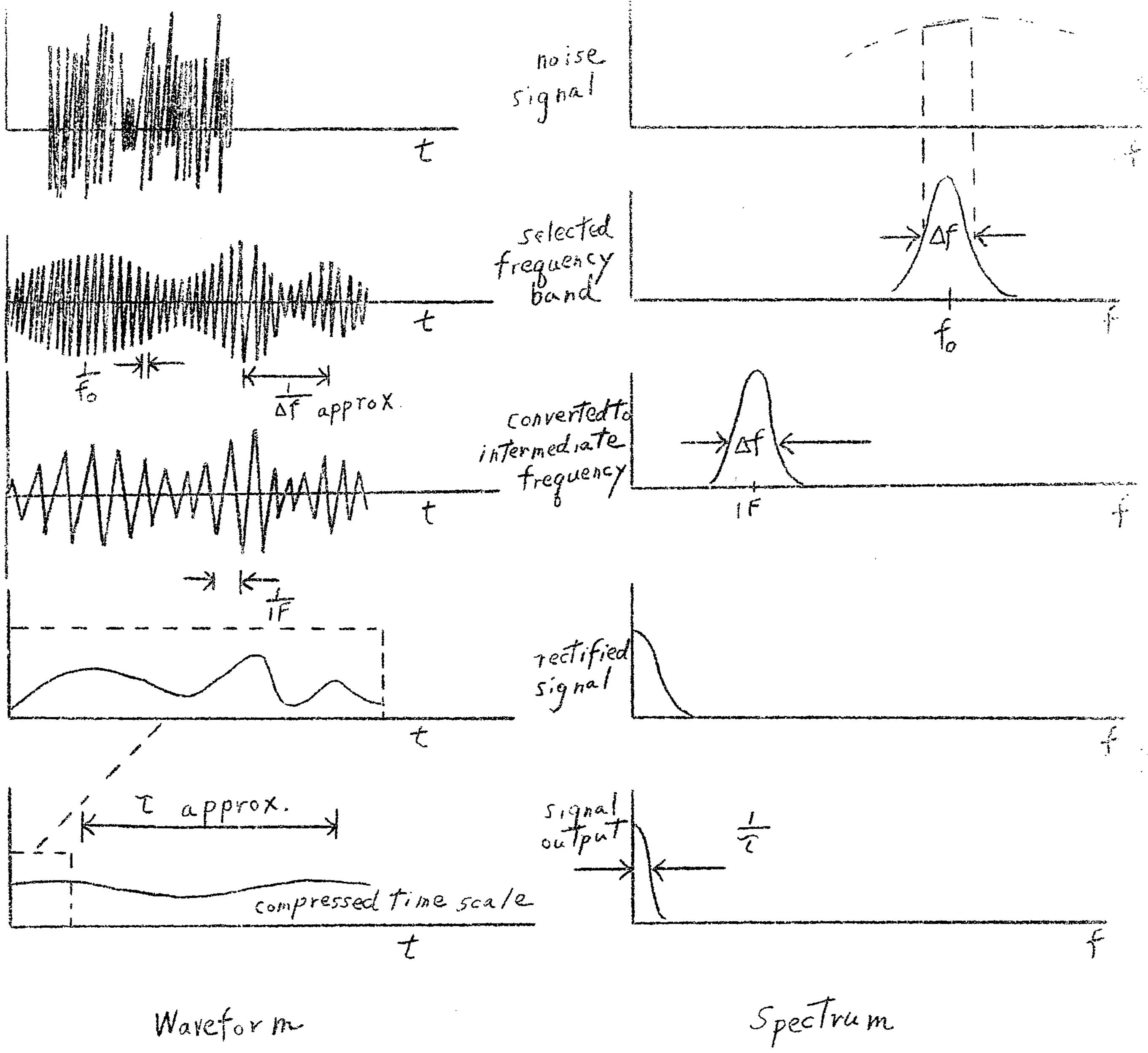

Figure 4. The changes in waveform and spectrum suffered by a noise signal in passing through a receiver. (from Pawsey and Bracewell).

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5. The Receiving System

Measurement of Aerial Temperature. The radio-frequency power delivered at the aerial output terminals is next delivered by a line to the receiver where the spectral components within the band Δf are selected and (in the usual heterodyne system) mixed with a local-oscillator signal to give a lower intermediate frequency resultant, which is amplified, rectified and fed into an output recorder whose reading corresponds to the aerial temperature. (See the graphical representation of Pawsey and Bracewell (1955) Figure 11, Section 2.4, **reproduced in Figure 4.**)

Now it is found that the radiation received from extraterrestrial objects has all the characteristics of noise (as does white light). Since the system is operating at the ambient temperature T_0, which is always greater than 0^oK, the thermionic tubes and resistors also generate noise. It is this factor, rather than the amplification of the system, that limits the theoretical sensitivity of the receiver. In fact, if g be the amplification factor of the system, i.e. the ratio of the power delivered to the input of the detector to the power available at the receiver input, then, neglecting line losses, the power delivered to the detector input is not gP but $g(P + P_r)$, where P_r represents the receiver noise, referred to the receiver input. The receiver noise may be specified by its noise temperature T_r, defined by the Nyquist relation

$$P_r = kT_r \Delta f . \qquad (1)$$

A more usual specification is in terms of the receiver noise factor N, which is defined as the ratio of total noise power in the actual receiver to the total noise power for a noise-free receiver when in each case the receiver is connected to an electrical load at the ambient temperature T_0. Then

$$N = \frac{P_o + P_r}{P_o} = \frac{T_0 + T_r}{T_0} , \qquad (2)$$

so that

$$P_r = (N-1)P_o = (N-1)kT_0\Delta f . \qquad (3)$$

For the present purpose it is cmnvestional to take $T_0 = 290^oK$. Now if P is the power available at the receiver input the recorder measures the power

$$P^* = P + P_r = k\left\{T_a + (N-1)T_0\right\} \Delta f . \qquad (4)$$

This is referred to the power $P_o^* = P_o + P_r$ when the receiver is connected to electrical load at the ambient temperature T_0. Then

$$P_o^* = Nk\, T_0\, \Delta f$$

and

$$P^* - P_o^* = k\,(T_a - T_0)\,\Delta f ,$$

so that

$$\frac{T_a - T_0}{T_0} = N\,\frac{P^* - P_o^*}{P_o^*} , \qquad (5)$$

giving the ratio T_a/T_0 in terms of the ratio of corresponding recorder measurements. Clearly, it is desirable in the interests of sensitivity to have N as close to the lower limit 1 as possible. In the meter wavelengths the figure N = 2 is realized, but at centimeter wavelengths the current best figures are between 7 and

The above analysis has been carried out without regard to losses in the line between the aerial output and the receiver input terminals. Now suppose only a fraction α of the aerial output is delivered to the receiver input terminals, the fraction $1 - \alpha$ being lost to the surroundings. Since the line is in equilibrium at the ambient temperature T_0, there must be the additional power $(1 - \alpha)\ P_o$ available at the receiver input terminals. Instead of (4) we now have

$$\begin{aligned} P^* &= \alpha P + (1-\alpha)\ P_o + P_r \\ &= k \left\{ \alpha T_a + (N-\alpha)\ T_0 \right\} \Delta f\ . \end{aligned} \tag{6}$$

Thus, to take account of line losses T_a and T_0 in (4) must be replaced by αT_a and αT_0, and N by N/α. It follows that (5) must be replaced by

$$\frac{T_a - T_0}{T_0} = \frac{N}{\alpha}\ \frac{P^* - P_o^*}{P_o^*}\ , \tag{7}$$

so that the recording sensitivity is reduced by the factor α.

<u>Sensitivity of Receiving System.</u> The chief characteristics of a receiving system are the amplification factor g, the noise factor N, the bandwidth Δf, and the overall time constant τ. The amplification factor is a matter of design, to ensure a large enough output. The noise factor has just been discussed. We go on to consider the limits imposed by inherent instrumental and and noise fluctuations.

The effect of the selection of a band of frequencies Δf is to introduce fluctuations* of period $\simeq 1/\Delta f$, whereas the effect of the finite time of response of the system to variations of power received is to redistribute the energy received in the rapid power fluctuations over a time $\simeq \tau$ (which is always $>> 1/f$). Thus variations within time intervals less than τ are effectively smoothed away. In particular, in order to minimize the effects of receiver bandwidth fluctuations it is necessary that we should have $\tau >> 1/\Delta f$.

Again, if noise from some external source is to be detectable against the background of receiver noise, it is necessary that the mean external noise power should be greater than the most probable fluctuation in receiver noise. This is determined by the bandwidth fluctuation rate Δf, which effectively allows Δf independent rectified noise contribtions per unit time, and the effective response time τ, during which there are $\tau \Delta f$ independent contributions. The **most probable RMS deviation ΔP_r from the** mean receiver noise intensity is a fraction of the order of $(\tau \Delta f)^{-1/2}$ of the mean P_r. The condition for detectability

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*Consider the simplest case where the receiver selects a band of frequencies $\Delta f = \Delta\omega/2\pi$, over which the amplitudes of the spectral components of the electric vector have the constant value $\underset{\sim}{A}$. Then the resultant field is the real part of

$$\underset{\sim}{E}_0 \, e^{-i\omega t} \Delta\omega = \int_{\omega - \Delta\omega/2}^{\omega + \Delta\omega/2} \underset{\sim}{A} \, e^{-i\omega t} \, d\omega$$

$$= \underset{\sim}{A} \, e^{-i\omega t} \, \frac{2 \sin(t \, \Delta\omega/2)}{t} \, .$$

Thus the mean amplitude is

$$\underset{\sim}{E}_0 = \underset{\sim}{A} \, \frac{\sin(\pi t \Delta f)}{\pi t \Delta f} \, .$$

of external noise of mean power P may then be expressed as

$$P > \Delta P_r = \beta P_r / \sqrt{(\tau \Delta f)} , \qquad (8)$$

where β is a constant of the order of unity.

It follows that receiver sensitivity may be increased by increasing the product $\tau \Delta f$. But this cannot be done without limit. In order to record phenomena such as solar bursts, which may last for only a few seconds, it is necessary that τ should be smaller than their decay time so as not to mask their decay. Again such phenomena often occur only on a narrow band of frequencies, so that Δf cannot be too large if this band is to be localized. The same considerations apply in detecting a small source against the background of sky noise. Thus, if a small contribution ΔP is to be detectable against the background noise $P + P_r$ from both the sky and the receiver, it is necessary that

$$\Delta P > \beta \frac{P + P_r}{\sqrt{(\tau \Delta f)}} . \qquad (9)$$

Expressed in terms of equivalent temperatures, (9) becomes

$$\Delta T_a > \beta \frac{T_a + (N-1) T_0}{\sqrt{(\tau \Delta f)}} . \qquad (10)$$

Thus it is more difficult to detect a faint source against the background of the Milky Way radiation than in the direction of the galactic poles.

Finally, we should note that in practice the limits to sensitivity are set not by background noise fluctuations so much as by fluctuations in g due to such factors as power-supply voltage fluctuations. This has led to the development of comparison systems in which the receiver is alternately switched from the aerial to a reference load (Dicke 1946, Ryle and Vonberg 1948).

1.6. The Aerial as a Radio Telescope

The term "radio telescope" has captured the imagination of the public in the same way as the term "electronic brain" is preferred to the more accurate term "electronic digital computer." The extent to which the name is appropriate can most readily be seen by listing the characteristics of a reflecting optical telescope and seeing the extent to which they can be applied to an aerial.

Apart from the effects of diffraction, when an optical telescope is directed at a certain part of the sky it collects light from the individual luminous objects, contained within a limited cone about the axis, in a concave mirror which in turn reflects the light into an optical system whence it can be directed on to such objects as a photographic plate, a photomultiplier, a spectrograph, or the eye of an observer. We list the following characteristics of the image:

(i) There is no magnification. The function of the mirror is simply to collect light and redirect it to the instruments that analyze and record the features being studied.

(ii) Individual objects in the cone of acceptance of the telescope are recorded as objects in a field of view, subject to the limitations of intensity of image, resolving power, and optical aberrations.

(iii) A wide band of frequencies and all polarizations of the incident radiation are accepted. The selection of a particular frequency band or polarization requires additional instruments.

The function of a radio receiving aerial is also to collect radio-frequency radiation from the sky for analysis. There is a "field of view" whose image exists in the focal plane. In fact only one point is "seen." All the radiation from within the cone of acceptance is integrated with different weights at the aerial feed, which is normally at the focus, into one signal at the aerial output, in accordance with the relations obtained in Sec. 1.4.

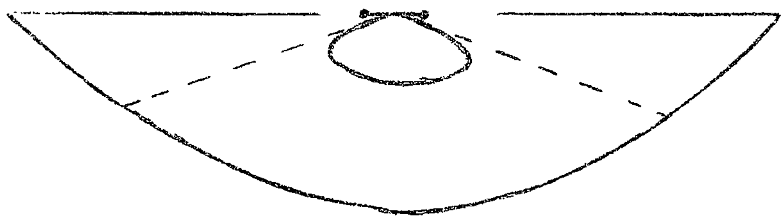
The aerial temperature T_a, corresponding to the single point "seen" in the field of view, is obtained as a weighted mean of the brightness temperatures of the rays over a region of the sky determined by the direction of orientation of the aerial. Moreover, only a small band of frequencies and only one polarization of radiation is accepted by any aerial.

It is clear, then, that the aerial together with the receiving and recording system would more appropriately be described as a frequency-selective radiometer. A close optical analogy would be obtained if the light from the reflector of a telescope were directed through a color filter and a polarizing system and on to the cathode of a small photoelectric cell placed at the focus. The output from the cell would then correspond to the output from an aerial. It would, of course, contain contributions from all directions of acceptance as given by the diffraction pattern of the telescope, which is the precise analogue of the aerial radiation pattern.

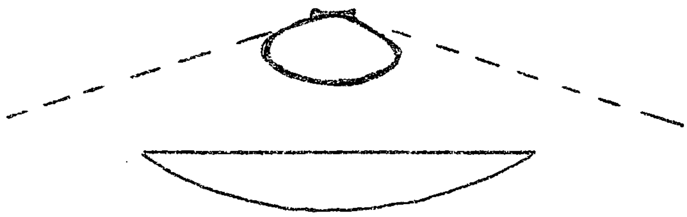
As in a telescope, the intensity of the signal produced by the radiation from any part of the sky can be increased by increasing the "aperture" and with it the effective area A of the aerial and the gain G. Again, just as the resolving power of a telescope is determined by diffraction at the edges of the iris stop and is proportional to the ratio λ/d, where d is the iris diameter, so is the aerial directional diagram - main and subsidiary lobes - determined by the overall diffraction or radiation pattern of the system. Because of the wavelength factor, an aerial of the same resolving power as a telescope would need an aperture of the order of 10^7 times as great. Again, the aperture of an optical telescope of the same resolving power as the Manchester 250-foot dish would be only 3×10^{-4} in.

Since an aerial with a fixed feed "sees" only one point in a field of view, a picture can be built up only by scanning an area of the sky, directing the aerial successively to different positions. Although it will take longer to cover a given area with a narrow-beam aerial, better resolution will be obtained than with a wider-beam aerial which smooths away some of the "fine structure" present in the distribution of the radiation over the sky.

The detection of faint radio sources, such as the external galaxies, against the general background of galactic radiation requires aerials of both large area (to enhance the signal) and large aperture (for resolution). Both requirements can be met only by aerials of large physical dimensions, i.e. of large aperture d. In telescopes the aperture d must not be so large that the off-axis effects of coma and astigmatism spoil the image. This is determined by the focal ratio f/d where f is the focal length of the mirror. This ratio specifies the angle of the cone of light from a parallel beam incident on the eyepiece or plate and, as we have seen, should not be too small if distortion is to be tolerable, e.g. a limiting



underfed reflector
$f/d = \frac{1}{4}$



overfed reflector
$f/d = \frac{1}{2}$

Figure 5. The effect of the choice of focal ratio

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figure of $f/d = 5$ has been quoted for the Newtonian focus (Cassegrain F/16, Coudé F/30). In aerials, the value of f/d is determined solely by the requirements of the "feed", the small receiving element which collects the radiation at the focus of the reflector. The simplest element is an elementary dipole for which, as we have seen, $G = \frac{3}{2} \cos^2 \theta$, and other dipole elements are not qualitatively different. The most efficient feed is a horn. A standard range of design figures is about $\frac{1}{4}$ to $\frac{1}{2}$ for f/d. Too high a figure gives too much "spill-over" resulting in contributions from the ground, whereas too low a figure results in too little weight being given to contributions from the outer parts of the reflector whence comes most of the collecting area.

For large-aperture telescopes there is also a mechanical upper limit on f/d imposed by the requirement of rigid support of the feed at the focus. The d = 250-feet Manchester paraboloid has $f/d = \frac{1}{4}$; the projected CSIRO d = 230-feet paraboloid has f/d = 0.4.

In order to scan an area of the sky these pencil-beam aerials need to be dirigible, and it is this feature which is responsible for all the design problems, in particular how to preserve the figure to sufficient accuracy in all positions under the stresses due to the variable distribution of weight and of wind loading.

These engineering difficulties have been avoided for a limited range of declinations by using fixed paraboloids and swinging the feed to various positions in the meridian plane. Then in each position a different strip of the sky passes through the beam as the earth rotates. By this means the feed scans the image field of the reflector in the focal plane. Hanbury Brown first used this technique at Jodrell Bank in 1949 with a suspended-wire aerial for which

d = 216 feet, $f/d = \frac{1}{2}$. In 1951 Bolton and his co-workers dug a paraboloid for which d = 80 feet, $f/d = \frac{1}{2}$ out of the sand at Dover Heights. The beam was swung some 14° on both sides of the vertical in the meridian plane. Unfortunately it is probable that the effect of 3rd-order coma (cf. Wolf 1951) becomes quite serious at such large angles of tilt. This may explain puzzling features centered at declination -45° in the 400 Mc/s survey of McGee, Slee and Stanley (1955) and effects ascribed by Hanbury Brown and Hazard (1953) to diffraction from the guy wires supporting the feed.

[illegible]. Aerial Smoothing - Elimination of Aerial Sensitivity Pattern

The question arises, given the results of a survey over the celestial sphere using a wide-beam aerial, i.e. the distribution of aerial temperature T_a, how far is it possible to reconstruct the distribution of brightness temperature T_b. We have Sec. 1.4 (11)

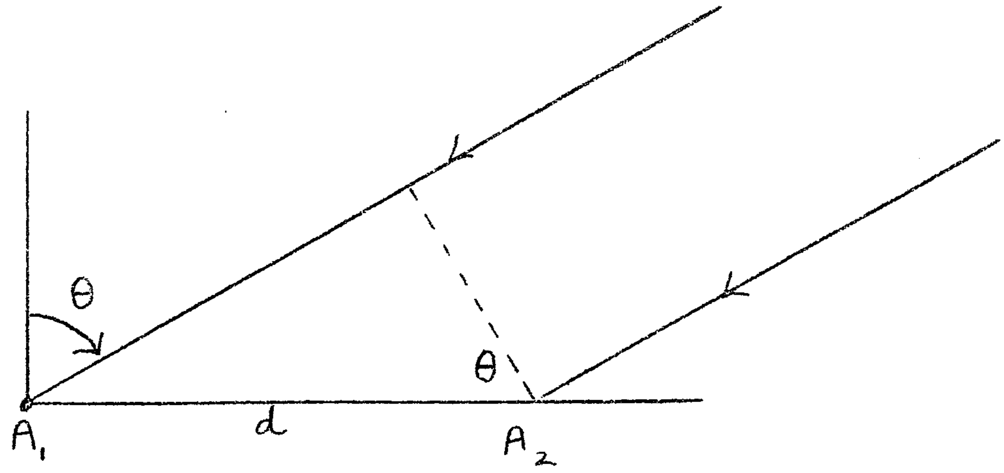
$$T_a(\underset{\sim}{n}) = \frac{1}{4\pi} \int G(\underset{\sim}{n},\underset{\sim}{n}') \; T_b \; (\underset{\sim}{n}) \; d\Omega(\underset{\sim}{n}') \; , \tag{1}$$

where T_a is known for all directions $\underset{\sim}{n}$ at which the aerial is pointed, and the gain function $G(\underset{\sim}{n},\underset{\sim}{n}')$ depends on the orientation of $\underset{\sim}{n}'$ relative to $\underset{\sim}{n}$. The equation (1) represents an integral equation for the determination of T_b, given T_a and the kernel G, so that our question is reduced to asking if a unique solution of the integral equation (1) is possible. The equation is linear and inhomogeneous. According to the theory of linear integral equations the general solution consists of what in the theory of linear differential equations is called the complementary function, i.e. the genera solution of the homogeneous equation

$$\int G(\underset{\sim}{n},\underset{\sim}{n}') \; T_b(\underset{\sim}{n}') \; d\Omega(\underset{\sim}{n}') = 0 \; , \tag{2}$$

together with a particular solution of (1). At first sight it might seem that the complementary function must be zero; since G is essentially a positive function, the distribution of T_b required to satisfy (2) must contain both positive and negative values, and therefore correspond to no real distribution. However, it is the value of the sum of the complementary function and the particular solution rather than the complementary function which is required to be positive everywhere, so that the difficulty is only superficial. The question has been discussed in some detail, for one-dimensional distributions by Bracewell and Roberts (1954) and for two-dimensional distributions by Burr (1955) and Bracewell (1956). It turns out that the complementary function consists of an arbitrary superposition of Fourier components of the distribution whose wave numbers are greater than the reciprocal of what is virtually the beamwidth. The result is plausible, since this component would be the first for which it is possible to have a zero average over the beamwidth. It follows that Fourier components of greater wave number, that may be present in the distribution of T_b, cannot contribute to T_a so that there is no means of restoring them by analysis of the T_a distribution. The "principal solution" of (1), defined as the distribution of T_b less the high wave-number components, may be obtained by replacing the integral equation (1) by a matrix equation for discrete values of T_a and G. (See e.g. Bolton and Westfold 1950.)

The practical conclusion to be drawn is that observations can never reveal fine structure of an order less than the aerial beamwidth. This feature has long been recognized in optical astronomy, in connection with the instrumental blurring of images and line profiles.



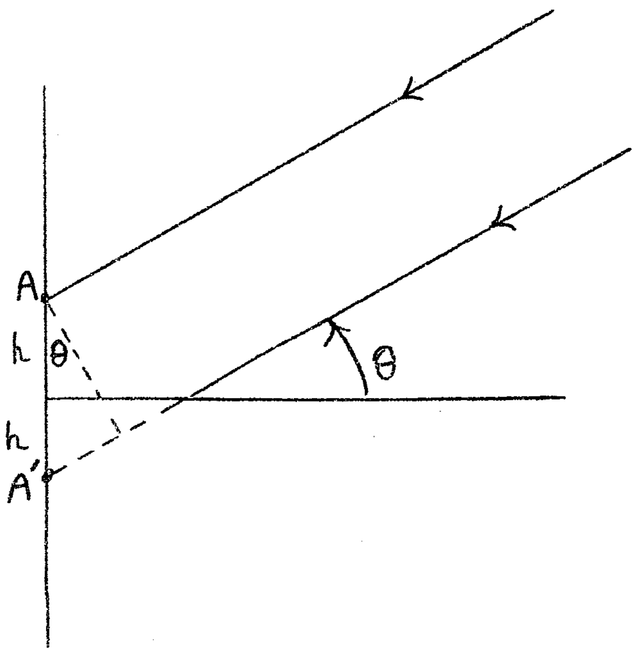


Figure 6. Path difference in a twin interferometer and a Lloyd's mirror

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8. Radio Interferometry

Simple theory. As in optics, the resolution of an instrument of given resolving power can be increased by utilizing the principles of interferometry. A receiving aerial of poor resolution when used in conjunction with another aerial (Michelson interferometer) or with its image (Lloyd's mirror), spaced many wavelengths apart, has its beam split into a number of interference fringes or lobes, distributed according to the spacing in wavelengths between the aerials.

Thus the response to a radio source of small angular dimensions passing through the main lobe of one of the aerials is to trace out the fringe pattern within the main lobe. When the angular width of the source is large, the interference fringes are overlapped and the fluctuations in the interference pattern of the source are reduced. The radio interferometer has proved invaluable in the detection of "point sources". If the particular interference fringe through which it is passing at any instant is known, one coordinate of its position can be determined. Its ~~strength~~ angular width can then be estimated from the ratio of the maximum to the minimum power received.

Consider two parallel rays from the same source, incident on two similarly oriented aerials distant d apart, and coming from a direction making an angle θ with the axis of the system. The phase difference between the two rays when incident on the aerials will be

$$\Delta = 2\pi d \sin \theta/\lambda \quad . \qquad (1)$$

(For a Lloyd's mirror at a height h above a plane reflecting surface

$$\Delta = 4\pi h \sin \theta/\lambda + \Delta' ,$$

where Δ' is the phase change suffered on reflection, usually taken as π.) If the outputs from the two aerials are connected to the same receiver, the available power from a point source is P , where

$$P = P_o (1 + \cos \Delta) \qquad (2)$$

and P_o is the available power when only one aerial is connected. To see this, consider a spectral component of the resultant electric vector of the radiation field incident on the interferometer. We may write this as the real part of the complex vector

$$\underset{\sim}{E} = \underset{\sim}{A}e^{-i\omega t} + \underset{\sim}{A}e^{-i(\omega t + \Delta)}$$

The mean power in the bandwidth Δf incident on the interferometer is proportional to

$$\overline{(R\ell E)^2}\Delta f = \frac{1}{2}|E|^2 \Delta f$$

$$= A^2 \Delta f(1 + \cos \Delta).$$

Since the mean power P_o incident on one element is proportional to $\frac{1}{2} A^2 \Delta f$, the mean incident power is P , where

$$P = 2 P_o (1 + \cos \Delta) .$$

The further factor $\frac{1}{2}$ in (2) results from the fact that the aerials are connected in parallel to the same receiver, so that the matched inpedance is half that required for a single aerial.

Now let the source be distributed over a finite angle, in such a manner that power $p(\phi)d\phi$ comes from the element $d\phi$ at angular distance ϕ from some central point on the source. Then, if this central point makes the angle θ_o with the interferometer axis, the available power from the source will be $P(\theta_o)$ where

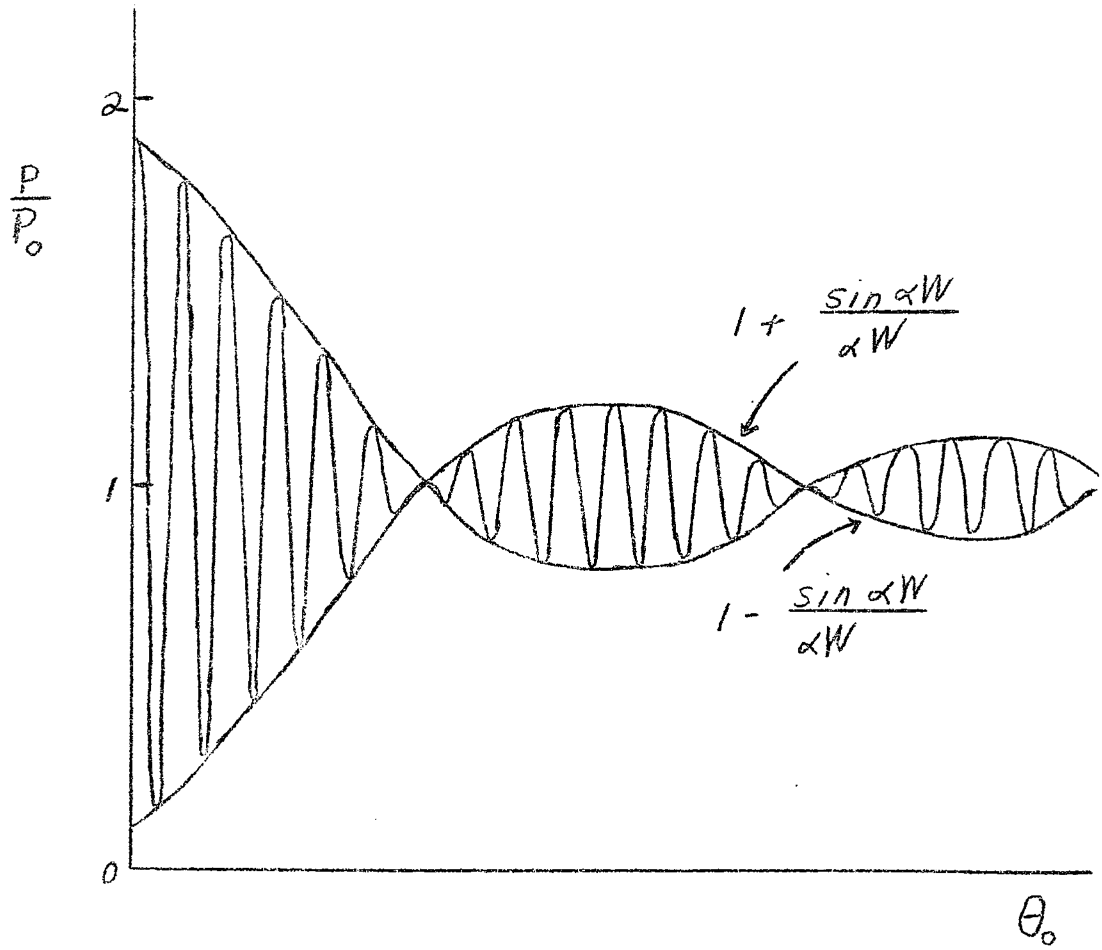

Figure 7. The effect of source width on the fringe pattern

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$$P(\theta_o) = \int p(\theta-\theta_o) \left\{1 + \cos \Delta(\theta)\right\} d\theta \quad . \tag{3}$$

If ϕ remains sufficiently small over the whole source, i.e. if the width 2W is sufficiently small, on making the substitution $\theta = \theta_o + \phi$, we may write for (1),

$$\Delta = \Delta_o + \alpha\phi \quad , \tag{4}$$

where

$$\left.\begin{aligned} \Delta_o &= \Delta(\theta_o) = (2\pi d/\lambda) \sin \theta_o \ , \\ \alpha &= (2\pi d/\lambda) \cos \theta_o \ , \end{aligned}\right\} \tag{5}$$

whence (3) becomes

$$P(\theta_o) = \int_{-W}^{W} p(\phi) \left\{1 + \cos (\Delta_o + \alpha\phi)\right\} d\phi \ . \tag{6}$$

If we neglect the variation of p over the source and replace it by its average $\bar{p}$ we can integrate (6) to get

$$P(\theta_o) = 2\bar{p}W \left(1 + \cos \Delta_o \frac{\sin \alpha W}{\alpha W}\right) \quad .$$

From a single aerial we should have had the power $P_o = 2\bar{p}W$. Hence we may write

$$P(\theta_o) = P_o \left(1 + \cos \Delta_o \frac{\sin \alpha W}{\alpha W}\right) \ , \tag{7}$$

which reduces to (2) for a point source. As the source moves through the interference fringes, θ_o and hence Δ_o and αW vary. It can be shown that if $\cot \theta_o >> W$ the fluctuations in Δ_o are more rapid than those in $(\sin \alpha W)/\alpha W$ so that P fluctuates between

$$P_{max} = P_o \left(1 + \frac{|\sin \alpha W|}{\alpha W}\right)$$

and

$$P_{min} = P_o \left(1 - \frac{|\sin \alpha W|}{\alpha W}\right) .$$

For a sufficiently narrow source the angular spacing between consecutive fringe maxima is

$$s = \lambda/(d \cos \theta_o) = 2\pi/\alpha , \tag{8}$$

which corresponds* to the period 2π of $\cos \Delta_o$. Thus, the power P_o from the source is given by

$$P_o = \frac{1}{2} (P_{max} + P_{min}) , \tag{9}$$

while the angular width 2W is determined by the relation

$$\frac{P_{max}}{P_{min}} = \frac{1 + |\sin \alpha W| / \alpha W}{1 - |\sin \alpha W| / \alpha W} . \tag{10}$$

As W increases the ratio P_{max}/P_{min} becomes closer to 1. For the particular freak case where the width 2W is sharply defined and equal to an integral number n of fringe spacings, (8) gives $\alpha W = n\pi$ and (7) gives a flat response as the source moves through the interference pattern, as might be expected for a source of uniform brightness. Otherwise, n is not integral and

$$\frac{P_{max}}{P_{min}} \sim 1 + \frac{1}{2n\pi} ,$$

for n sufficiently large.

A similar obscuring effect on the interference pattern is due to the finite bandwidth Δf of the radiation received. For this

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* $\Delta(\theta_o + s) - \Delta(\theta_o) \doteq (2\pi d/\lambda) s \cos \theta_o = 2\pi$.

see Bolton and Slee (1953). The fringe spacings for neighboring wavelengths differ and the patterns overlap, so that the resultant maxima and minima are less sharply defined.

Fourier Analysis of the Source Distribution in One Dimension

The relation (6) can be used to infer the distribution $p(\phi)$ of the source if α is regarded as a variable. In general, the source boundaries will not be distinct, but the contribution from outside some angle W may be disregarded or estimated by extrapolation.

In either case the limits of integration may be taken as $\pm \infty$, so that

$$P(\theta_o) - P_o = \int_{-\infty}^{\infty} p(\phi) \, (\cos \Delta_o \cos \alpha\phi - \sin \Delta_o \sin \alpha\phi) d\phi.$$

The right-hand side represents the sum of a cosine and a sine transform of the distribution $p(\phi)$. If it can be assumed that $p(\phi)$ is an even function, i.e. that the distribution is symmetrical, the sine transform will be zero and we shall have

$$\frac{P(\theta_o) - P_o}{2 \cos \Delta_o} = \int_0^{\infty} p(\phi) \cos \alpha\phi \, d\phi$$

$$= \bar{p}(\alpha), \tag{11}$$

the Fourier cosine transform of $p(\phi)$. The function $\bar{p}(\alpha)$ can be found empirically by taking observations for different α which will give $\left\{P(\theta_o) - P_o\right\} / 2 \cos \Delta_o$. Then the distribution $p(\phi)$ can be inferred by evaluating the dual transform

$$p(\phi) = \frac{2}{\pi} \int_0^{\infty} \bar{p}(\alpha) \cos \alpha\phi \, d\alpha. \tag{12}$$

It can be seen from (5) that $\overline{p}(\alpha)$ could, in theory, be obtained by taking the readings of P as the center of the source moves through different positions θ_c. However, records of radiation received are not so clear that the source contribution can be confidently estimated against the general background. The difference $P_{max} - P_{min}$, however, is practically independent of the background and can more easily be estimated from the records for successive maxima and minima, at which $\cos \Delta_o = \pm 1$ while θ_o remains practically constant. If this difference is determined for different spacings d at the same position θ_o, the function

$$\overline{p} = (P_{max} - P_{min})/4 \qquad (13)$$

will be determined for different values of α given by (5). The distribution $p(\phi)$ in one dimension is then given by (12). In practice it is not necessary to wait for a source to drift through the interference pattern. A steady variation in phase Δ can be introduced electrically into the system so that the fringes sweep across the source instead.

Such a strip analysis was first used by Stanier (1950) and again by Machin (1951) on the Sun at 500 and 81.5 Mc/s. The resulting function $p(\phi)$ represents the sum of contributions over the chord ϕ = constant. Assuming that the distribution possesses circular symmetry they were able to derive the brightness distribution across any diameter. The method was also used by Bolton, Westfold, Stanley and Slee (1954) to determine the brightness distribution along a discrete source of large angular width.

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R A D I O   A S T R O N O M Y

Chapter 2

Electromagnetic Wave Propagation in an Ionized Gas

CHAPTER 2

Electromagnetic Wave Propagation in an Ionized Gas

2.1. Basic Equations

The equations of the electromagnetic field represent the Faraday-Neumann law

$$\text{curl } \underset{\sim}{E} + \frac{\partial \underset{\sim}{B}}{\partial t} = 0 \qquad (1)$$

and Maxwell's generalization of Ampère's law

$$\text{curl } \underset{\sim}{H} - \frac{\partial \underset{\sim}{D}}{\partial t} = \underset{\sim}{J} , \qquad (2)$$

where $\underset{\sim}{J}$ is the total current density and $\underset{\sim}{E}$, $\underset{\sim}{B}$, $\underset{\sim}{H}$, $\underset{\sim}{D}$ the field vectors. In an ionized gas

$$D = \epsilon_v \underset{\sim}{E} , \quad \underset{\sim}{H} = \frac{1}{\mu_v} \underset{\sim}{B} , \qquad (3)$$

where μ_v, ϵ_v are the permeability and permittivity of free space. If the gas consists of various species for which the molecules have mass m_s, charge e_s, number density n_s and velocity v_s,

$$\underset{\sim}{J} = \sum n_s e_s \underset{\sim}{v}_s .$$

These velocities are referred to the mass velocity $\underset{\sim}{v}$ of the fluid, which is defined by the relation

$$(\sum n_s m_s)\underset{\sim}{v} = \sum n_s m_s \underset{\sim}{v}_s .$$

The mean peculiar velocity is then defined as the difference

$$\underset{\sim}{V}_s = \underset{\sim}{v}_s - \underset{\sim}{v} ,$$

so that

$$\underset{\sim}{J} = (\sum n_s e_s)\underset{\sim}{v} + \sum n_s e_s \underset{\sim}{V}_s .$$

The quantity

$$\rho_e = \sum n_s e_s$$

is the density of free charge so that first term $\rho_e \underset{\sim}{v}$ is the convection current density. The second term represents the flow of current relative to the mass of fluid; this is the conduction current

$$\underset{\sim}{j} = \sum n_s e_s \underset{\sim}{V}_s .$$

Thus we have the total current as the sum of convection and conduction currents

$$\underset{\sim}{J} = \rho_e \underset{\sim}{v} + \underset{\sim}{j} . \tag{4}$$

Since the total charge is conserved $\underset{\sim}{J}$ and ρ_e are subject to the equation of continuity

$$\frac{\partial \rho_e}{\partial t} + \operatorname{div} \underset{\sim}{J} = 0 . \tag{5}$$

The subsidiary Maxwell equations follow from taking the divergence of (1) and of (2) using (5)

$$\operatorname{div} \underset{\sim}{B} = 0 \tag{6}$$

and

$$\operatorname{div} \underset{\sim}{D} = \rho_e . \tag{7}$$

We shall here be interested in the case where there is no mass motion of the gas so that $\underset{\sim}{v} = 0$ and hence

$$\underset{\sim}{J} = \underset{\sim}{j} . \tag{8}$$

The total current is now the conduction current which is subject to an equation which is a generalization of Ohm's law

$$\underset{\sim}{j} = \sigma \underset{\sim}{E} ,$$

where σ is the conductivity. It is obtained by considering the equations of motion of each species and combining them so as to give a condition on $\underset{\sim}{j}$. Fortunately, the gas is Lorentzian, i.e. the molecules of one species, viz. the electrons, are considerably lighter and hence more mobile than all the others. Consequently, it is a good approximation to regard the conduction current as being principally due to electrons, of mass m, charge e and mean peculiar velocity $\underset{\sim}{V}$.

Since $\underset{\sim}{j}$ is principally due to electrons, we may obtain the generalized Ohm's law from a consideration of the motion of an average electron induced by the electromagnetic field. We have

$$m\dot{\underset{\sim}{V}} = e\,(\underset{\sim}{E} + \underset{\sim}{V}\wedge\underset{\sim}{B}) - \nu m\underset{\sim}{V} ,$$

where the last term represents the damping effect due to collisions with the other molecules of the gas. Roughly speaking, the average effect of a collision is to extinguish the momentum of an electron, so that if the electron suffers on the average ν collisions per unit time it loses momentum at the rate $\nu m\underset{\sim}{V}$. A more refined free-path argument is given by Burkhardt (1950). In terms of $\underset{\sim}{j} = ne\underset{\sim}{V}$, we have the required result

$$\frac{\partial \underset{\sim}{j}}{\partial t} + \nu \underset{\sim}{j} = \frac{ne^2}{m} \underset{\sim}{E} + \frac{e}{m} \underset{\sim}{j}\wedge\underset{\sim}{B} . \qquad (9)$$

The matter has been investigated for a binary ionized gas by the more precise velocity-distribution methods of Chapman and Enskog by Westfold (1953). The damping constant ν turns out to be a mean collision frequency given in terms of the coefficient of mutual diffusion D and the kinetic temperature T of the gas,

$$\nu = \frac{n_i}{n_i + n} \frac{kT}{mD} , \tag{10}$$

where n_i is the number density of the other component of the binary gas and k is Boltzmann's constant.

For a fully ionized gas Chapman and Cowling (1953) Sec. 10.33 give

$$D = \frac{3}{16(n+n_i)} \left(\frac{2\ kT}{\pi m}\right)^{1/2} \left(\frac{8\pi\ \epsilon_v kT}{ee_i}\right)^2 / \ln(1+\bar{\upsilon}_{01}^{\ 2}) , \tag{11}$$

where $\bar{\upsilon}_{01}$ is a constant whose value will be discussed in Sec. 3.3. Thus we have

$$\nu = \frac{4}{3} n_i \left\{\frac{\pi}{2\ m(kT)^3}\right\}^{1/2} \left(\frac{e\ e_i}{4\pi \epsilon_v}\right) \ln\ (1 + \bar{\upsilon}_{01}^{\ 2}) , \tag{12}$$

which is 4/3 times the electron collision frequency.

Again, for a slightly ionized gas the encounters made by the electrons are principally with the neutral atoms. In this case, Chapman and Cowling (1953) Secs. 9.81, 10.22 give

$$D = \frac{3}{16(n+n_i)a^2} \left(\frac{2\ kT}{\pi m}\right)^{1/2} , \tag{13}$$

where πa^2 is the cross section for an encounter. Thus

$$\nu = \frac{8}{3} n_i \left(\frac{2\pi\ kT}{m}\right)^{1/2} a^2 . \tag{14}$$

We note, incidentally, that by identifying the two formulae for D we get an effective collision cross section for inverse-square-law encounters.

[In c.g.s. e.m.u. $\mu_v = 4\pi$, $\epsilon_v = 1/4\pi c^2$.]

2.2 Lorentz Theory

We now consider the propagation of electromagnetic radiation in an ionized medium at rest under no other external fields, in which the pressure is uniform. We neglect such mass velocities and pressure gradients as may be impressed on the medium by the electromagnetic field, so that the generalized Ohm's law in the form Sec. 2.1(9) is applicable

$$\frac{\partial \underset{\sim}{j}}{\partial t} + \nu \underset{\sim}{j} = \frac{ne^2}{m}\underset{\sim}{E} + \frac{e}{m}\, j \wedge B \;.$$

If the impressed electromagnetic field is small, the last term is of the second order and may be neglected. Thus we get the linear conductivity relation

$$\frac{\partial \underset{\sim}{j}}{\partial t} + \nu \underset{\sim}{j} = \frac{ne^2}{m}E \;. \qquad (1)$$

Clearly, the conductivity is a function of the frequency of the radiation so that the medium is dispersive. In practice we are usually interested in only a narrow band of frequencies so we can consider a monochromatic field represented by the real parts of the complex vectors

$$\underset{\sim}{E} = \underset{\sim}{E}' e^{-i\omega t} \text{, etc.} \qquad (2)$$

Then (1) becomes

$$(-i\omega + \nu)\underset{\sim}{j} = \epsilon_\nu \omega_o^2 \underset{\sim}{E}$$

where we have written

$$\omega_o^2 = \frac{ne^2}{m\epsilon_\nu}, \tag{3}$$

[In c.g.s. units $f_o^2 = 8.06 \times 10^7 n \text{ sec}^{-2}$, $f_o = 8.98 \times 10^3 \sqrt{n}$ c/s, where $f_o = \omega_o/2\pi$.]

which we shall see in Sec. 2.4 is the square of the angular electron plasma frequency. Thus

$$\underset{\sim}{j} = \sigma \underset{\sim}{E} \tag{4}$$

where*

$$\sigma = i\omega\epsilon_\nu \frac{x}{1 + iz} \tag{5}$$

and

$$x = \omega_o^2/\omega^2, \quad z = \nu/\omega . \tag{6}$$

The dimensionless quantities x,z were first introduced by Appleton (1932) and are now standard. Taking (4) with Sec. 2.1(2) and (3) we have

$$\text{curl } \underset{\sim}{B} + i\omega\mu_\nu\epsilon_\nu \underset{\sim}{E} = \mu_\nu \sigma \underset{\sim}{E}$$

i.e.

$$\text{curl } \underset{\sim}{B} + i\omega\mu_\nu\epsilon_\nu \left(1 - \frac{x}{1 + iz}\right)\underset{\sim}{E} = 0 ,$$

so that the quantity

$$K = 1 - \frac{x}{1 + iz} \tag{7}$$

is effectively the complex dielectric constant of the medium. With Sec. 2.1(1) we now have the two equations

---

* Note that (3) reduces to Drude's formula $\sigma = ne^2/m\nu$ in the static case where $\omega = 0$.

$$\text{curl}\ \underset{\sim}{B} + \frac{i\omega}{c^2} K\underset{\sim}{E} = 0 \tag{8}$$

and

$$\text{curl}\ \underset{\sim}{E} - i\omega\ \underset{\sim}{B} = 0\ , \tag{9}$$

to determine the complex monochromatic field vectors. Taking the divergence of each we get

$$K\ \text{div}\ \underset{\sim}{E} = \text{div}\ \underset{\sim}{B} = 0\ , \tag{10}$$

which correspond to Sec. 2.1(6) and (7). Now in general, ω, v and n will not be such that K = 0, so we shall for the present take

$$\text{div}\ \underset{\sim}{E} = 0\ , \tag{11}$$

whence, by Sec. 2.1(7) and (3),

$$\rho_e = \epsilon_v\ \text{div}\ \underset{\sim}{E} \tag{12}$$

is also zero. We shall see in Sec. 2.4 that in the other case we have an oscillatory distribution of free charge density ρ_e and an electromagnetic field which is not propagated.

Eliminating $\underset{\sim}{B}$ from (8) and (9) we get

$$\text{curl curl}\ \underset{\sim}{E} - \frac{\omega^2}{c^2} K\ \underset{\sim}{E} = 0\ ,$$

which by virtue of (11) reduces to

$$(\nabla^2 + \frac{\omega^2}{c^2} K)\underset{\sim}{E} = 0\ . \tag{13}$$

The simplest solutions are those corresponding to plane waves

$$\underset{\sim}{E} = \underset{\sim}{E}''\ e^{-i(\omega(t - q\ n\cdot\underset{\sim}{r}/c)}\ , \tag{14}$$

in which $\underset{\sim}{n}$ is a unit vector, in the direction of propagation and q is the complex refractive index, written

$$q = \mu + i\chi \, , \tag{15}$$

where μ is the ordinary real refractive index and χ the absorption index, such that $\omega\chi/c$ is the attenuation factor and

$$\kappa = 2\omega\chi/c \tag{16}$$

the absorption coefficient. Note that we distinguish between the amplitude attenuation factor and the energy absorption coefficient. The latter is more important in radiation theory, where we are concerned with the transfer of energy.

The complex refractive index is now determined by substitution from (14) into (13). We get

$$\frac{{}^{2}}{c^2}(-q^2 + K)\underset{\sim}{E} = 0$$

whence

$$q^2 = K \, , \tag{17}$$

the same formal result as for propagation in a dielectric medium. Also, by (11),

$$\underset{\sim}{E} \cdot \underset{\sim}{n} = 0 \tag{18}$$

and by (9)

$$\underset{\sim}{B} = \frac{q}{c} \, \underset{\sim}{n} \wedge \underset{\sim}{E} \, , \tag{19}$$

so that the field is entirely transverse to the direction of propagation.

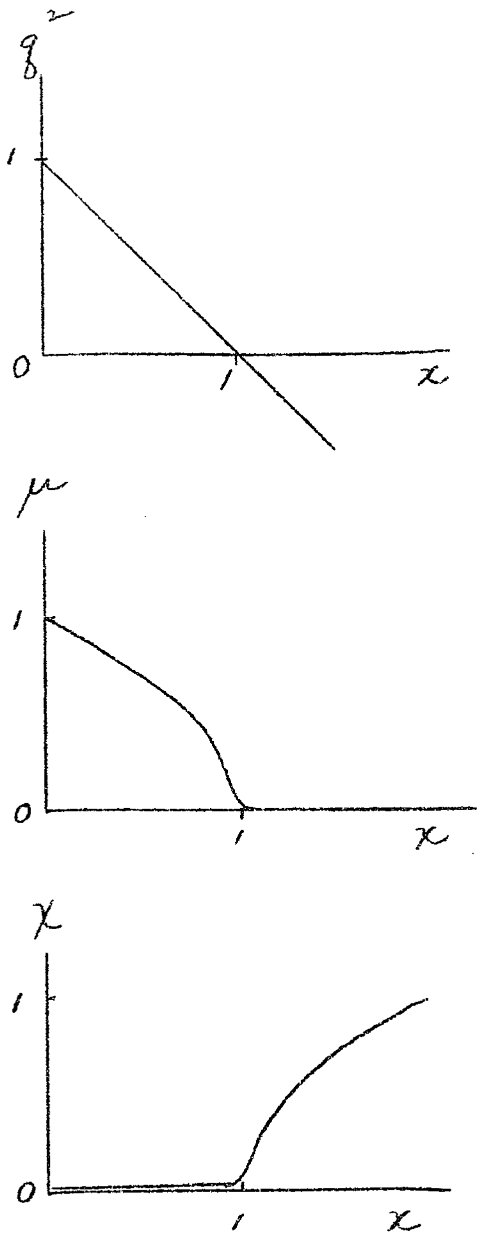

Figure 1. $q^2(z=0)$, μ, and χ in Lorentz theory

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Refractive Index and Absorption Coefficient. The equations (7) and (17) give the complex refractive index in terms of the parameters of the medium

$$q^2 = 1 - \frac{x}{1 + iz} . \qquad (20)$$

In the case where collisional effects are negligible, $q^2 = 1 - x$ is wholly real. This is the formula of Eccles (1912) and Larmor (1924). There are two distinct cases, separated by the zero $x = 1$, viz.

$0 \leqslant x \leqslant 1$, whence $q^2 \geqslant 0$ so that, by (15),

$$\mu = \sqrt{(1 - x)}, \quad \chi = 0 ;$$

and $x \geqslant 1$, whence $q^2 \leqslant 0$ so that

$$\mu = 0, \quad \chi = \sqrt{(x - 1)} .$$

In the former case there is no absorption and the wave proceeds with the phase velocity $c/\sqrt{(1 - x)}$. In the latter case, there is heavy absorption and the wave has an infinite phase velocity. The behavior is exhibited in Figure 1. Thus, if a wave of given frequency ω is propagated in a medium where the electron density n is such that $\omega_0{}^2 = ne^2/m\epsilon_v \leqslant \omega^2$, there is no absorption and the wave proceeds without inhibition; if $\omega_0{}^2 > \omega^2$ propagation is inhibited. For the practical cases of propagation in the ionosphere and the corona z is small, but not negligible, so that first approximations to the equations for μ and χ in (20) are, since

$$\mu^2 - \chi^2 = 1 - \frac{x}{1 + z^2} , \quad 2\mu\chi = \frac{zx}{1 + z^2} ,$$

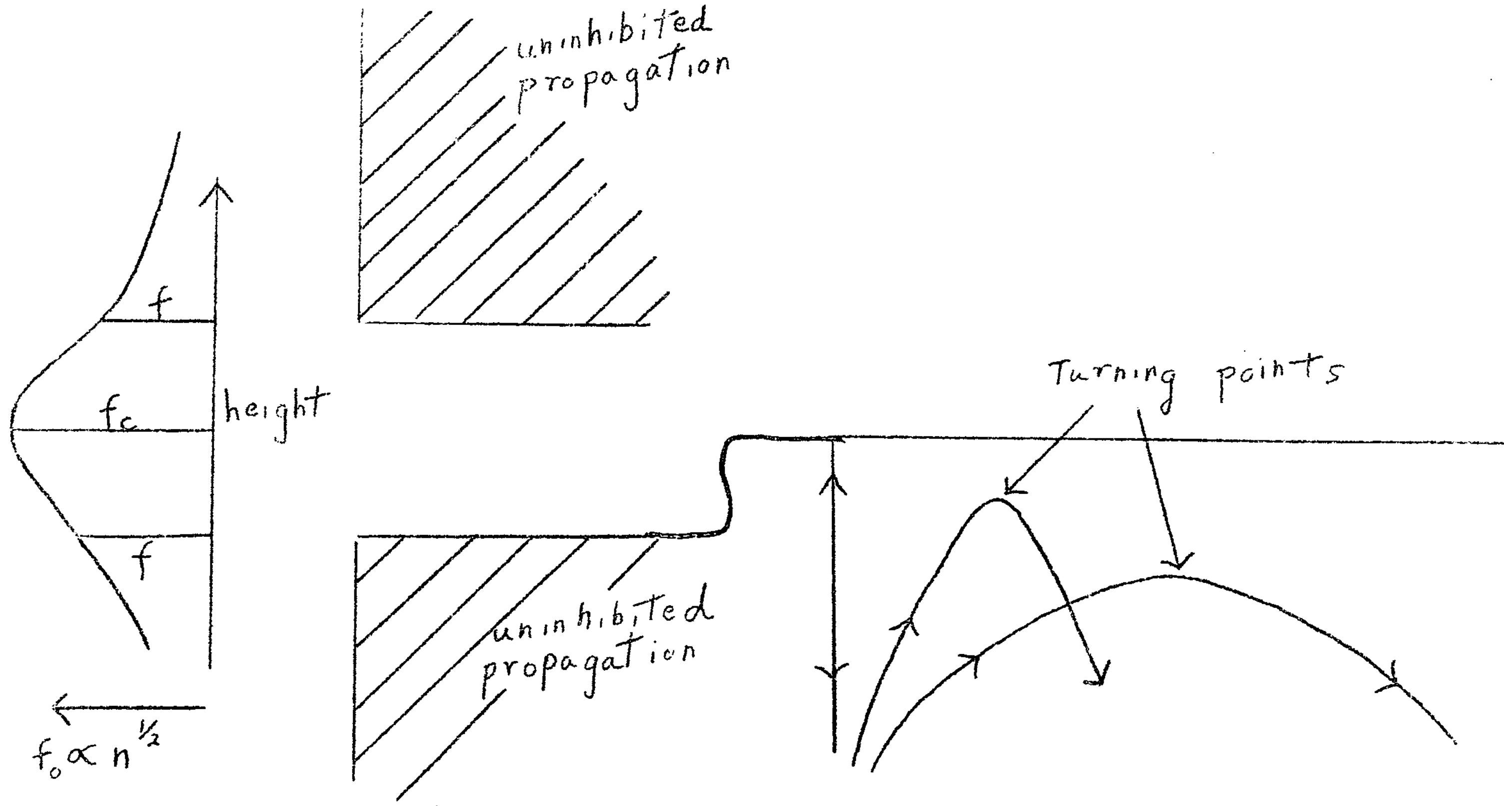

Figure 2(a). Propagation in an ionospheric layer for $f_c >$

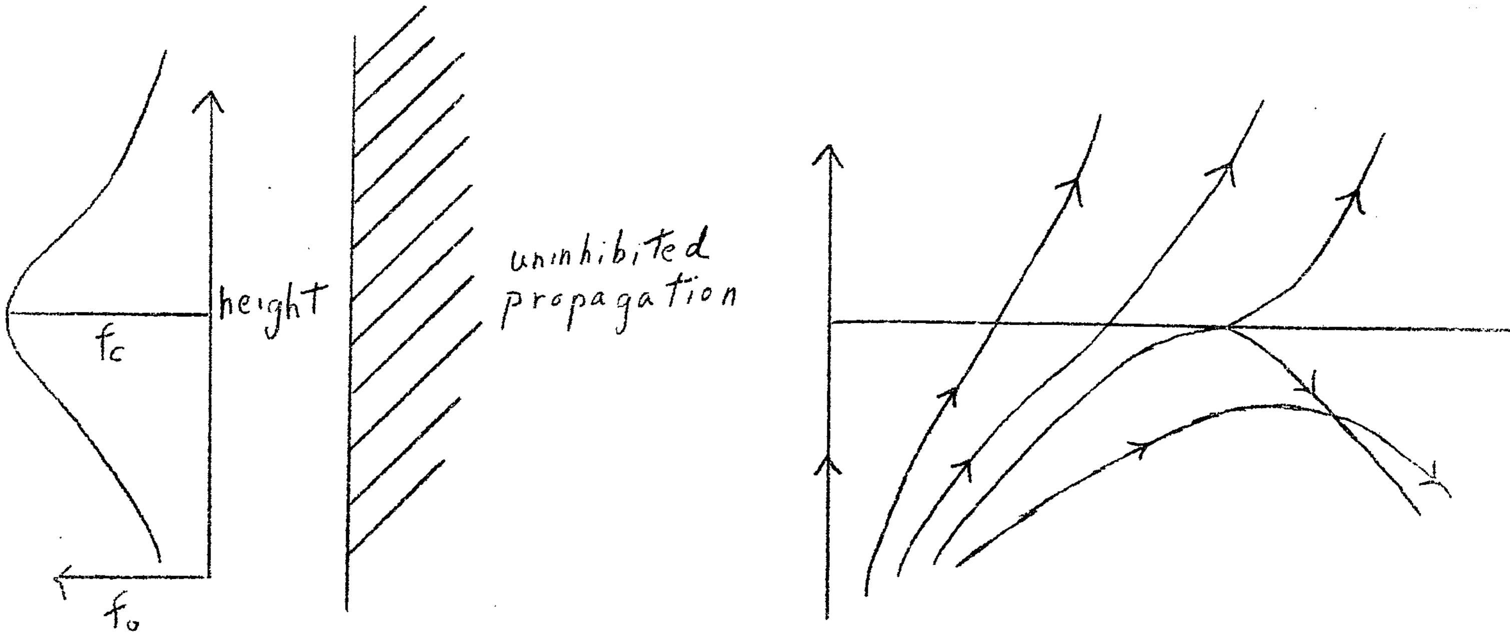

Figure 2(b). Propagation in an ionospheric layer for $f_c < f$.

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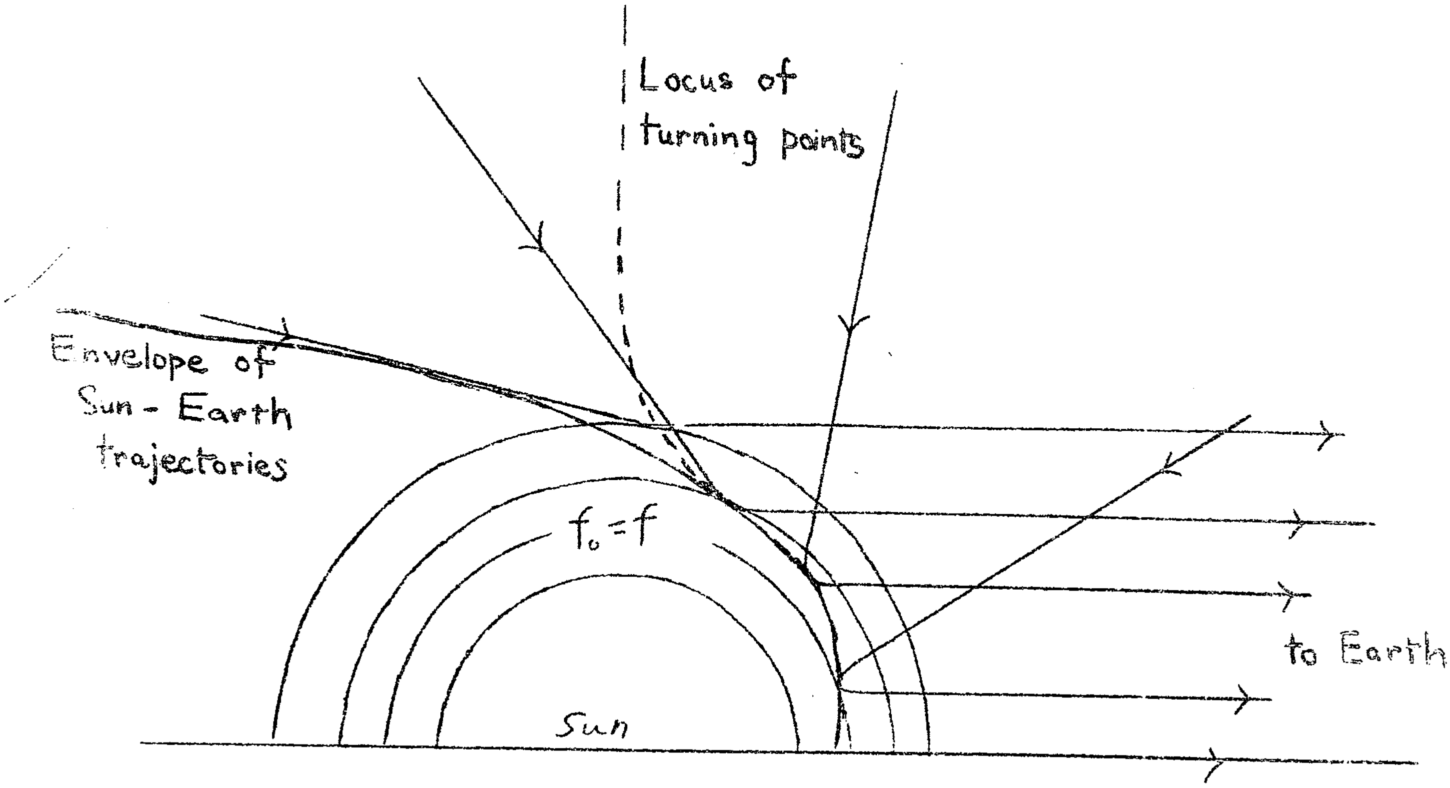

Figure 3. Propagation in the solar atmosphere.

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$$\left.\begin{array}{lll} \mu = \sqrt{(1-x)}\ , & \chi = zx/2\sqrt{(1-x)}\ , & 0<x<1, \\ \mu = \chi = \sqrt{(z/2)}\ , & & x = 1, \\ \mu = zx/2\sqrt{(x-1)}\ , & \chi = \sqrt{(x-1)}\ , & x>1. \end{array}\right\} \quad (2$$

Here there is only slight absorption for $0<x<1$ so that propagation is relatively uninhibited; for $x>1$ there is again heavy absorption and the phase velocity is large, but not infinite. The above results, obtained for uniform media, may be applied to non-uniform media so long as μ and χ are substantially constant over a medium wavelength. The formula (14) is then replaced by

$$\underset{\sim}{E} = \underset{\sim}{E}'' \, e^{-i\omega(t-\int q\,ds/c)} \ ,$$

where the integral is taken over a ray trajectory.

The criterion that $0<x<1$ for relatively uninhibited propagation restricts the regions of the ionosphere and the solar atmosphere accessible to observation on any frequency ω to those for which $\omega_0^2 < \omega^2$, i.e. to the lower and outer layers. The equivalent path and absorption for ray trajectories in the ionosphere have been calculated by Jaeger (1947) and in the corona by Burkhardt and Schlüter (1949) and Jaeger and Westfold (1950), using Snell's law. The situation is illustrated in Figures 2(a), 2(b) and 3, in terms of the frequencies $f = \omega/2\pi$, $f_0 = \omega_0/2\pi$. The critical frequency of an ionospheric layer is $f_c = (f_0)_{max}$. Typical values are $f_c = 3$, 4.5 and 7.5 Mc/s in the E, F_1 and F_2 layers. Note that only one trajectory of the family appropriate to an observer reaches the level $f_0 = f$. The accessible regions are delineated by the envelope of the family of trajectories.

Average Energy Density and Poynting Flux. In a monochromatic field the energy density U and Poynting flux vector $\underset{\sim}{S}$ are given by

$$\left.\begin{aligned} U &= \frac{1}{2}\epsilon_v(R\ell E)^2 + \frac{1}{2\mu_v}(R\ell B)^2 , \\ \underset{\sim}{S} &= R\ell \underset{\sim}{E} \wedge R\ell \underset{\sim}{H} . \end{aligned}\right\} \tag{22}$$

Since we are only interested in variations over times greater than the period it is sufficient to consider averages over the period. Thus

$$\bar{U} = \frac{1}{4}\epsilon_v|E|^2 + \frac{1}{4\mu_v}|B|^2 ,$$

$$\bar{\underset{\sim}{S}} = \frac{1}{2} R\ell\,(\underset{\sim}{E}\wedge\underset{\sim}{H}^*) = \frac{1}{2} R\ell\,(\underset{\sim}{E}^*\wedge\underset{\sim}{H}) ,$$

where an asterisk denotes the complex conjugate. Since, by (19),

$$\frac{1}{\mu_v}|B|^2 = \frac{|q|^2}{\mu_v c^2}|E|^2 = |q|^2\epsilon_v|E|^2 ,$$

$$\underset{\sim}{E}^*\wedge\underset{\sim}{H} = \frac{q}{\mu_v c}\underset{\sim}{E}^*\wedge(\underset{\sim}{n}\wedge\underset{\sim}{E}) = q\,\epsilon_v c|E|^2\,\underset{\sim}{n} ,$$

we get, using (16)

$$\left.\begin{aligned} \bar{U} &= \frac{1}{4}\epsilon_v(1+\mu^2+\chi^2)|E''|^2\,e^{-\kappa\underset{\sim}{n}\cdot\underset{\sim}{r}} , \\ \bar{S} &= \frac{1}{2}\epsilon_v\mu c|E''|^2\,e^{-\kappa\underset{\sim}{n}\cdot\underset{\sim}{r}}\,\underset{\sim}{n} . \end{aligned}\right\} \tag{23}$$

Both $\bar{U}$ and $\bar{S}$ are reduced according to the absorption coefficient κ as the wave proceeds. Moreover,

$$\bar{\underset{\sim}{S}} = \frac{2\mu}{1+\mu^2+\chi^2}\,\bar{U}\,c\,\underset{\sim}{n} , \tag{24}$$

so that energy density is propagated with the velocity

$2\mu c/(1 + \mu^2 + \chi^2)$ in the direction of propagation. This is less than c, as it should be.

Group Velocity. Another velocity appropriate to the propagation of a group of waves clustered about the frequency ω is the group velocity $u = d\omega/dk$ where the propagation constant $k = \omega\mu/c$. This gives

$$u = \frac{c}{d(\omega\mu)/d\omega} . \qquad (25)$$

In a region of uninhibited propagation (21) and (6) give

$$\omega\mu = \sqrt{(\omega^2 - \omega_0^2)}$$

so that

$$\frac{d(\omega\mu)}{d\omega} = \frac{\omega}{\sqrt{(\omega^2 - \omega_0^2)}} = \frac{1}{\sqrt{(1 - x)}} . \qquad (26)$$

Hence

$$u = \mu c, \qquad (27)$$

which is again less than c, as it should be.

2.3. The Magneto-ionic Theory

This theory was devised by Appleton (1925) and independently by Nichols and Schelleng (1925) to account for propagation in the ionosphere under the influence of the Earth's magnetic field. The situation is the same as for the Lorentz theory, except that in addition a uniform magnetic field of induction B_0 is imposed on the medium. In this case $B_0 \gg B$/the magnitude of the magnetic vector of the field so that the Ohm's-law relation Sec. 2.1(9) becomes

$$\frac{\partial \underset{\sim}{j}}{\partial t} + \underset{\sim}{\omega}_B \wedge \underset{\sim}{j} + \nu \underset{\sim}{j} = \epsilon_v \omega_0^2 \underset{\sim}{E} , \qquad (1)$$

where we have written

$$\underset{\sim}{\omega}_B = e\underset{\sim}{B}_0/m \, , \tag{2}$$

{for B_0 in gauss $f_B = \omega_B/2\pi = 2.80\, B_0 \times 10^6$ c/s} a vector whose magnitude is the angular electron gyro-frequency; it is the angular velocity with which an electron rotates about a line of force of the field $\underset{\sim}{B}_0$. The equation (1) is again linear homogeneous, but the new term has introduced an anisotropy into the conductivity relation. For monochromatic fields we have a tensor relation between the complex vectors,

$$(-i\omega + \nu)\underset{\sim}{j} + \underset{\sim}{\omega}_B \wedge \underset{\sim}{j} = \epsilon_\nu \omega_0^2 \underset{\sim}{E} \, ,$$

or

$$(1 + iz)\, \underset{\sim}{j} + i\, \underset{\sim}{y} \wedge \underset{\sim}{j} = i\omega \epsilon_\nu X\, E \, , \tag{3}$$

where we have now introduced Appleton's third dimensionless parameter

$$\underset{\sim}{y} = \underset{\sim}{\omega}_B/\omega \, . \tag{4}$$

We can solve for $\underset{\sim}{j}$ in terms of $\underset{\sim}{E}$ by considering components $\underset{\sim}{E}_\parallel$, $\underset{\sim}{E}_\perp$ parallel and perpendicular to $\underset{\sim}{B}_0$ the imposed static field. We find

$$\underset{\sim}{j} = i\omega\epsilon_\nu X \left\{ \frac{1}{1 + iz} \underset{\sim}{E}_\parallel + \frac{1 + iz}{(1 + iz)^2 - y^2} \underset{\sim}{E}_\perp - \frac{i}{(1 + iz)^2 - y^2} \underset{\sim}{y} \wedge \underset{\sim}{E} \right\} . \tag{5}$$

Comparing (5) with Sec. 2.2(4) we see that the conductivity along lines of force is the same as the Lorentz value; there is also a component of $\underset{\sim}{j}$ along the direction of $\underset{\sim}{E}_\perp$, and in addition a component transverse to both B_0 and $\underset{\sim}{E}$, the Hall current. As for static fields, we may write (cf. Cowling 1953)

$$\underset{\sim}{j} = \sigma \underset{\sim}{E}_{\parallel} + \sigma^{I} \underset{\sim}{E}_{\perp} + \sigma^{II} \underset{\sim}{k} \wedge \underset{\sim}{E} , \tag{6}$$

where* σ is given by Sec. 2.2(5),

$$\left.\begin{aligned} \sigma^{I} &= \frac{(1 + iz)^2}{(1 + iz)^2 - y^2} \sigma , \\ \sigma^{II} &= \frac{iy(1 + iz)}{(1 + iz)^2 - y^2} \sigma , \end{aligned}\right\} \tag{7}$$

and $\underset{\sim}{k}$ is a unit vector in the direction of $\underset{\sim}{B}_0$. Since the charge on an electron is negative the last definition implies $\underset{\sim}{y} = -y\underset{\sim}{k}$. This result is equivalent to the tensor relation

$$\underset{\sim}{j} = \underset{\approx}{\sigma} \cdot \underset{\sim}{E} , \tag{8}$$

which replaces Sec. 2.2(4). Instead of Sec. 2.2(8) and (7), we get

$$\text{curl } \underset{\sim}{B} + i\frac{\omega}{c^2} \underset{\approx}{K} \cdot \underset{\sim}{E} = 0 , \tag{9}$$

where $\underset{\approx}{K}$ is the complex dielectric tensor, given in terms of $\underset{\approx}{\sigma}$ and the unit tensor $\underset{\approx}{U}$ by

$$\underset{\approx}{K} = \underset{\approx}{U} - \frac{\underset{\approx}{\sigma}}{i\omega\epsilon_0} . \tag{10}$$

Then elimination of $\underset{\sim}{E}$ from the field equations gives

$$\text{curl curl } \underset{\sim}{E} - \frac{\omega^2}{c^2} \underset{\approx}{K} \cdot \underset{\sim}{E} = 0 ,$$

or

$$(\nabla^2 - \text{grad div})\underset{\sim}{E} + \frac{\omega^2}{c^2} \underset{\approx}{K} \cdot \underset{\sim}{E} = 0 . \tag{11}$$

---

* Note that (6) reduces to the standard static formulae $\sigma^{I} = \frac{\nu^2}{\nu^2 + \omega_B^2}\sigma$, $\sigma^{II} = \frac{\nu\,\omega_B}{\nu^2 + \omega_B^2}\sigma$ for $\omega = 0$.

We note that in this case div $\underset{\sim}{E} \neq 0$ so that, by 2.1(7) and (3), there is a distribution of free charge ρ_e. Taking the divergence of (9) we see, however, that

$$\text{div } (\underset{\approx}{K} \cdot \underset{\sim}{E}) = 0 , \tag{12}$$

as in crystalline media.

Again the simplest solutions of (11) are plane-wave solutions of the type 2.2(14). In these, since div $\underset{\sim}{E} \neq 0$, the electric vector $\underset{\sim}{E}$ is not transverse to the direction of propagation $\underset{\sim}{n}$, although by (12) the vector $\underset{\approx}{K} \cdot \underset{\sim}{E}$ is; by 2.2(9) so is the magnetic vector $\underset{\sim}{B}$. For such solutions (11) gives

$$-q^2(\underset{\sim}{E} - \underset{\sim}{n}\,\underset{\sim}{n} \cdot \underset{\sim}{E}) + \underset{\approx}{K} \cdot \underset{\sim}{E} = 0 , \tag{13}$$

a linear homogeneous relation between the components of $\underset{\sim}{E}$ that can be satisfied only if q satisfies the determinantal equation

$$\left| q^2(\underset{\approx}{U} - \underset{\sim}{n}\,\underset{\sim}{n}) - \underset{\approx}{K} \right| = 0 . \tag{14}$$

Clearly, q will depend on the direction of propagation. Then the relations between the components of $\underset{\sim}{E}$ are specified by (13); hence also for $\underset{\sim}{j}$ and $\underset{\sim}{B}$. Thus in a magneto-ionic medium both the complex refractive index and the polarization of plane-wave fields are determined by the imposed magnetic field and the direction of propagation. It is not arbitrary as in a Lorentz medium.

The algebra involved in the programme indicated is rather heavy, but some simplification follows if, as in crystal optics, we choose axes along the principal directions of the tensor $\underset{\approx}{K}$.

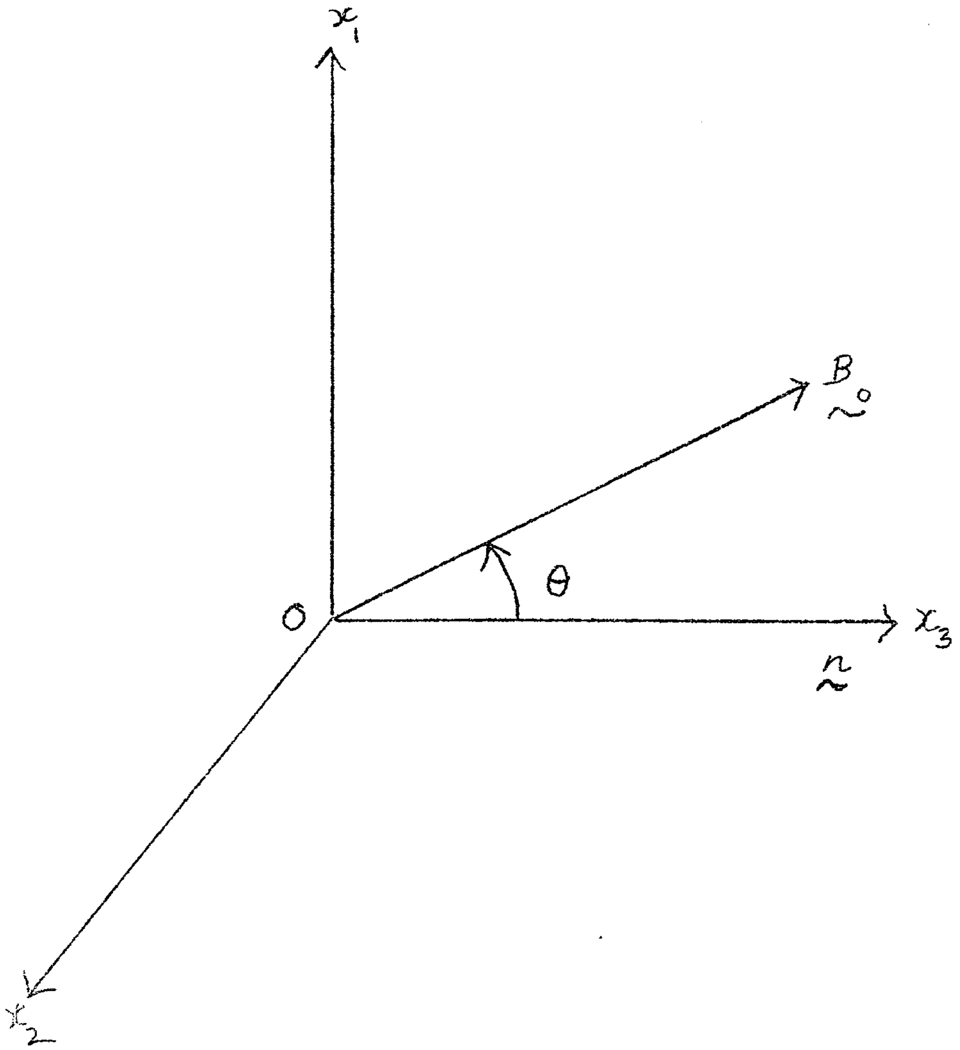

Figure 4. Cartesian axes for the magneto-ionic theory

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This was done by Westfold (1949); since the reference triad of unit vectors is complex the procedure is a little sophisticated.

Again, if only one wave is being considered in a uniform medium it may be convenient to choose the direction of propagation as one of the coordinate axes and the plane containing $\underset{\sim}{n}$ and $\underset{\sim}{B}_o$ as a coordinate plane. This will be useful when considering the polarization of the wave. However, the various components of $\underset{\approx}{K}$ make for complicated algebra.

The simplest procedure appears to be to work from (3) instead of the more complicated inverse relation (8). Thus, we eliminate $\underset{\sim}{E}$ and $\underset{\sim}{B}$ and are left with a simpler relation between the components of $\underset{\sim}{j}$. We choose Cartesian axes such that $\underset{\sim}{n} = (0,0,1)$ and $\underset{\sim}{B}_o$ is parallel to the plane Ox_1x_3 such that $\underset{\sim}{B}_o = B_o(\sin\theta, 0, \cos\theta)$, i.e., θ is the angle $(\underset{\sim}{n}, \underset{\sim}{B}_o)$. Then $\underset{\sim}{y} = -y(\sin\theta, 0, \cos\theta)$. Writing the ratio of the transverse components of $\underset{\sim}{j}$ as

$$\frac{j_2}{j_1} = Q, \tag{15}$$

and carrying out the elimination, we get

$$q^2 = 1 - \frac{x}{1 + iz + iyQ\cos\theta} \tag{16}$$

where

$$Q^2 - \frac{iy\sin^2\theta/\cos\theta}{1 + iz - x}Q + 1 = 0. \tag{17}$$

These are together equivalent to the Appleton-Hartree formula. Since (17) is quadratic there are only two possible types of wave for any $\mathbf{n}$. The direction of $\mathbf{j}$ associated with either of these is given by

$$\frac{j_1}{1} = \frac{j_2}{Q} = \frac{j_3}{iyQ \sin\theta / (1 + iz - x)} . \qquad (18)$$

To determine the polarizations of the waves we need to find the corresponding relation between the components of $\mathbf{E}$. From (3) and (18) we get

$$\frac{E_1}{1} = \frac{E_2}{Q} = \frac{E_3}{\dfrac{ixyQ \sin\theta}{(1 + iz - x)(1 + iz + iyQ \cos\theta)}} . \qquad (19)$$

The ratio of the components of $\mathbf{E}$ in the transverse plane is equal to the ratio of the same components of $\mathbf{j}$, i.e.,

$$\frac{E_2}{E_1} = \frac{j_2}{j_1} = Q .$$

Thus Q is the complex polarization which determines the characteristics of the ellipse traced out by the projection of $\mathbf{E}$ onto the transverse plane. Since again

$$\mathbf{B} = \frac{q}{c}\, \mathbf{n} \wedge \mathbf{E} , \qquad (20)$$

$\mathbf{B}$ is entirely transverse, such that

$$\frac{B_1}{Q} = \frac{B_2}{-1} = \frac{B_3}{0} . \qquad (21)$$

Particular cases

(i) Longitudinal propagation ($n \parallel B_o$, $\sin\theta = 0$).

(17): $Q^2 + 1 = 0$, whence $Q = \pm i$, specifying $\left.\begin{matrix}RH\\LH\end{matrix}\right\}$ circular polarization

(16): $$q^2 = 1 - \frac{x}{1 + iz \mp y|\cos\theta|/\cos\theta} .$$

(19): $$\frac{E_1}{1} = \frac{E_2}{\pm i} = \frac{E_3}{0} .$$

The fields of both waves lie entirely in the transverse plane. The two values of q^2, corresponding to

$$Q = \pm i|\cos\theta|/\cos\theta ,$$

are

$$\left.\begin{aligned} q^2 &= K_1 = 1 - \frac{x}{1 + iz - y} , \\ q^2 &= K_2 = 1 - \frac{x}{1 + iz + y} . \end{aligned}\right\} \qquad (22)$$

K_1 and K_2 are two of the principal values of the complex dielectric tensor K, corresponding to the complex eigenvectors perpendicular to B_o (see Westfold, 1949).

The characteristics of propagation are similar to those of the Lorentz formula Sec.2.2(20). For $z = 0$ the zeros of K_1, K_2 are at $x = 1 \mp y$, respectively. Instead of Sec.2.2(21) we have

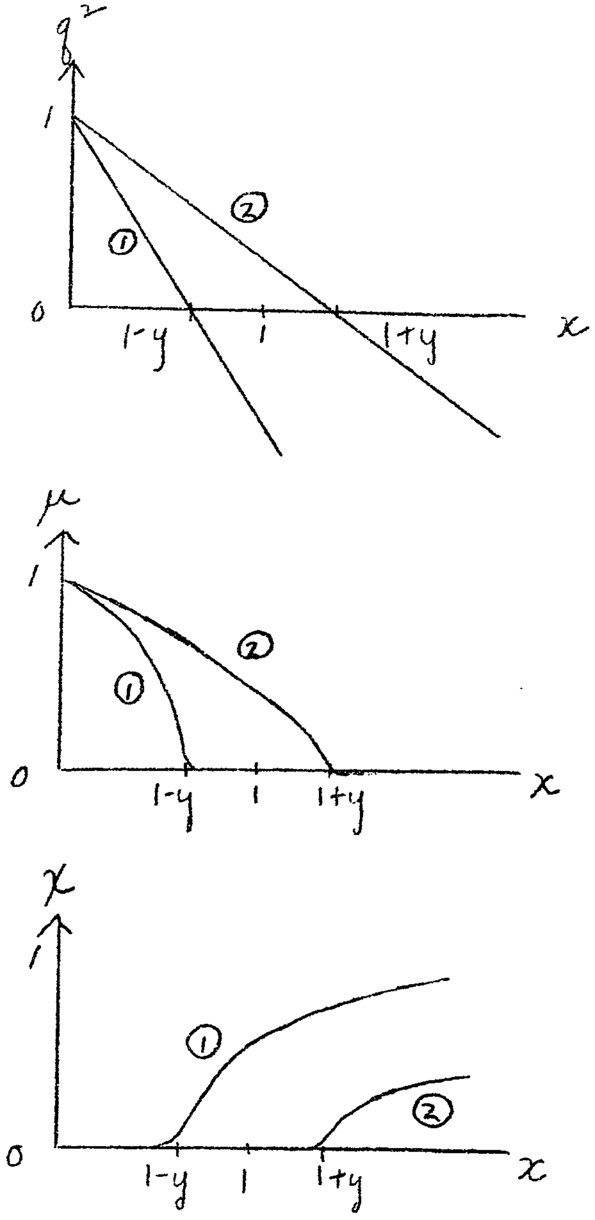

Figure 5. $q^2(z=0)$, μ, χ for waves 1 and 2.

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$$
\left.
\begin{aligned}
\mu &= \sqrt{\left(1 - \frac{x}{1 \mp y}\right)}, \quad \chi = \frac{zx}{(1 \mp y)^2} \Big/ 2\sqrt{\left(1 - \frac{x}{1 \mp y}\right)}, && 0 < x < 1 \mp y, \\
\mu &= \chi = \sqrt{z/2(1 \mp y)}, && x = 1 \mp y, \\
\mu &= \frac{zx}{(1 \mp y)^2} \Big/ 2\sqrt{\left(\frac{x}{1 \mp y} - 1\right)}, \quad \chi = \sqrt{\left(\frac{x}{1 \mp y} - 1\right)}, && x > 1 \mp y .
\end{aligned}
\right\} \quad (23)
$$

The regions of relatively uninhibited propagation for wave 1 is $0 < x < 1 - y$, and for wave 2 it is $0 < x < 1 + y$.

(ii) <u>Transverse propagation ($n \perp B_0$, $\cos\theta = 0$).</u>

<u>To 1st order for cos θ small:</u>

(17): $Q = \dfrac{iy/\cos\theta}{1 + iz - x} \Rightarrow (0)$, or $\dfrac{1 + iz - x}{iy/\cos\theta} \rightarrow (\infty)$, specifying linear polarization.

(16): $q^2 = 1 - \dfrac{x}{1 + iz - y^2/(1 + iz - x)}$, or $1 - \dfrac{x}{1 + iz}$.

(19): $\dfrac{E_1}{0} = \dfrac{E_2}{1} = \dfrac{E_3}{\dfrac{ixy}{(1 + iz)(1 + iz - x) - y^2}}$, or $\dfrac{E_1}{1} = \dfrac{E_2}{0} = \dfrac{E_3}{0}$.

The two values of q^2, corresponding to

$$Q = (\infty), (0),$$

are

$$
\left.
\begin{aligned}
q^2 &= K_3 = 1 - \frac{x}{1 + iz}, \\
q^2 &= K_4 = 1 - \frac{x}{1 + iz - y^2/(1 + iz - x)} .
\end{aligned}
\right\} \quad (24)
$$

K_3 is the third principal dielectric constant which corresponds to the real eigenvector parallel to B_0. In the transverse plane the locus of the projection of E is linear; this is actually the locus

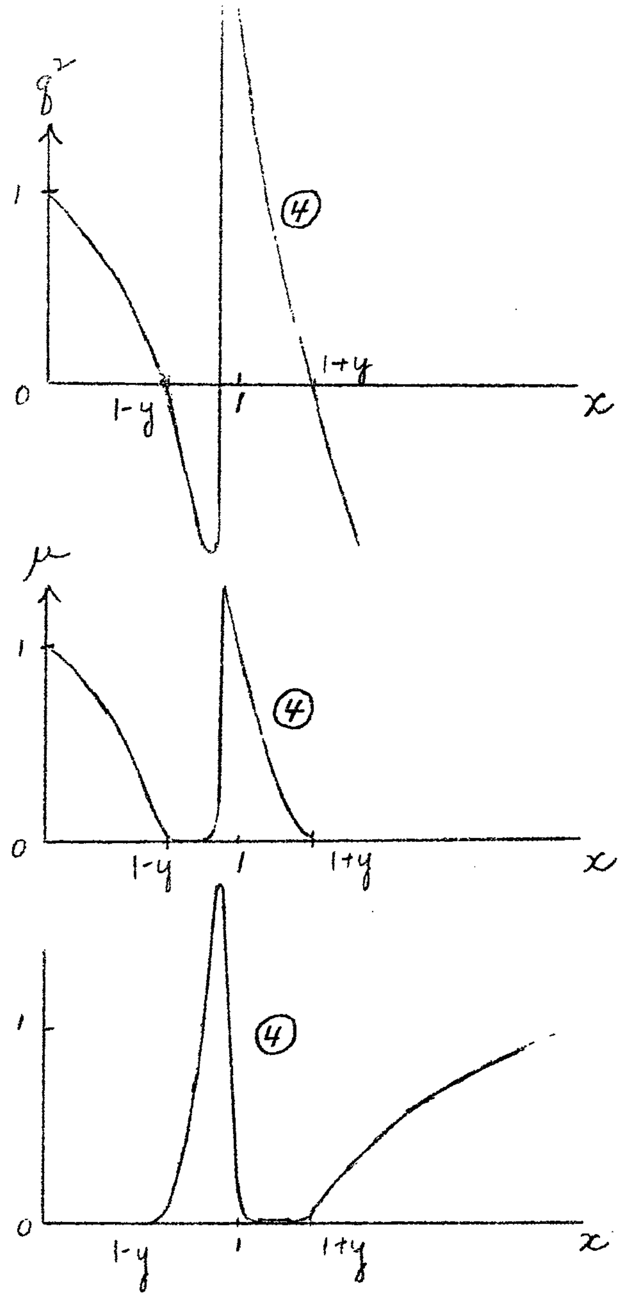

Figure 6. $q^2(z=0)$, μ, χ for wave 4
(For wave 3 see Figure 1).

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of the electric vector of wave 3, its direction being always parallel to $\underset{\sim}{B}_o$, whereas for wave 4, while the projection of $\underset{\sim}{E}$ is perpendicular to $\underset{\sim}{B}_o$, there is also a component in the direction of propagation. The propagation characteristics of wave 3 have been investigated in Sec. 2.2. For $z = 0$, $K_4 = 0$ where $x = 1 \mp y$, the same zeros as for K_1 and K_2. However, in addition, K_4 has an infinity where $x = 1 - y^2$ (see Figure 6). If the effect of collisions is small, but not negligible, the infinity is rounded off in the usual manner of resonance curves, crossing the x-axis between large positive and negative values. The curves for μ, χ, depict two regions of relatively uninhibited propagation, $0 \leqslant x \leqslant 1 - y$ and $1 - y^2 \leqslant x \leqslant 1 + y$. The latter, more dense region is usually inaccessible because of the barrier between $x = 1 - y$ and $x = 1 - y^2$.

General direction of propagation

There is an continuous transition in the properties of the two waves as the direction of propagation changes from longitudinal ($\theta = 0^o$) to transverse ($\theta = 90^o$) to longitudinal ($\theta = 180^o$) in the opposite sense. This is somewhat complex but it has been traced by Westfold (1951) for the case $0 \leqslant y \leqslant 1$ illustrated above and when $y \geqslant 1$. Generally speaking, there are two waves which we denote by α, β having complementary elliptic polarizations Q_α, Q_β such that $Q_\alpha Q_\beta = 1$.

For a range of directions near $\theta = 0^o$ and $\theta = 180^o$ the characteristics are similar to those of longitudinal propagation and are termed quasi-longitudinal (QL). The wave α, like wave 2, has a zero in $q^2(z = 0)$ at $x = 1 + y$ and the wave β at $x = 1 - y$, like wave 1. Thus, there is relatively uninhibited propagation of

$$\text{wave } \alpha \text{ for } 0 \leqslant x \leqslant 1 + y,$$
$$\text{wave } \beta \text{ for } 0 \leqslant x \leqslant 1 - y.$$

The characteristics of propagation change over to the quasi-transverse (QT) form at critical directions such that $Q_\alpha = Q_\beta$, $q_\alpha = q_\beta$. By (17) this is where both $x = 1$ and $z = y \sin^2\theta/2 |\cos\theta|$. If, as is usual, $z \ll y$ the QL zones are within a small angle θ from the directions of the field $\underset{\sim}{B}_0$ and of $-\underset{\sim}{B}_0$, where $\theta^2 = 2\nu/\omega_B$. For larger values of sin θ the wave α, like wave 3, has a zero in $q^2(z = 0)$ at $x = 1$ and the wave β zeros at $x = 1 \mp y$ and an infinity at $x = (1 - y^2)/(1 - y^2\cos^2\theta)$, like wave 4. Thus, there is relatively uninhibited propagation of

wave α for $0 \leqslant x \leqslant 1$,

wave β for $0 \leqslant x \leqslant 1 - y$ and $(1 - y^2)/(1 - y^2\cos\theta) \leqslant x \leqslant 1 + y$ if $0 \leqslant y \leqslant 1$,

and for $0 \leqslant x \leqslant 1 + y$ if $y \geqslant 1$.

For $0 \leqslant y \leqslant 1$ the latter, more dense, region is usually inaccessible to observation.

Although it has not been possible* to calculate reliable trajectories, equivalent path, and absorption for rays in the ionosphere and corona taking into account the effects of an imposed

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*See Westfold (1949).

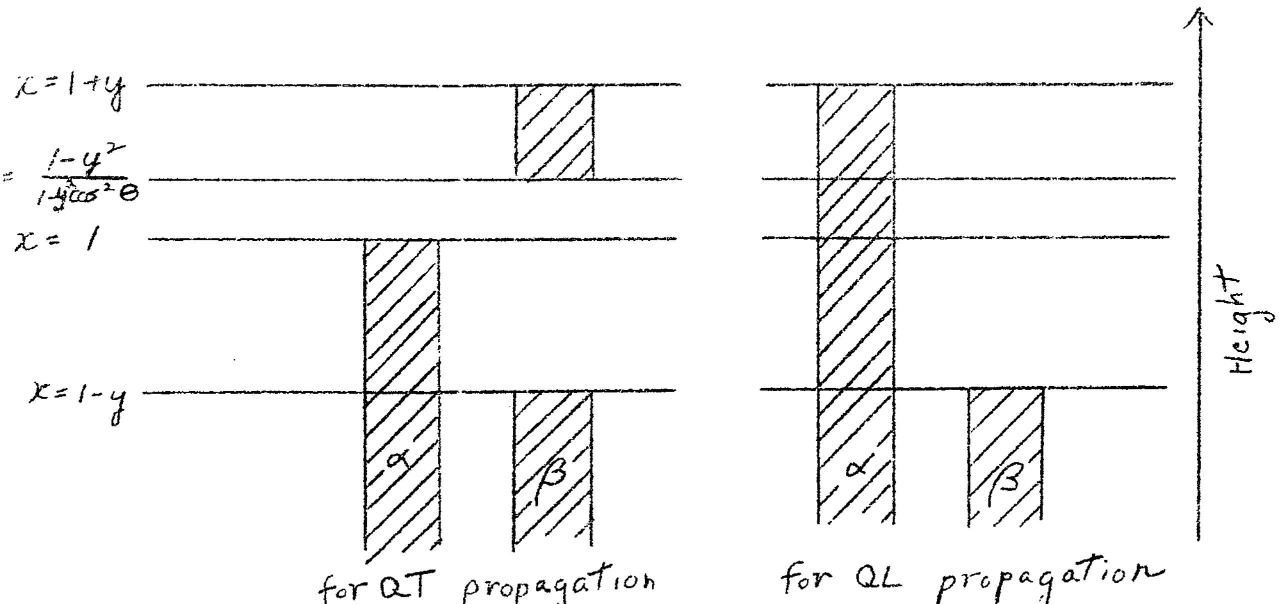

Figure 7. Regions of uninhibited propagation in an ionospheric layer (lower region only).

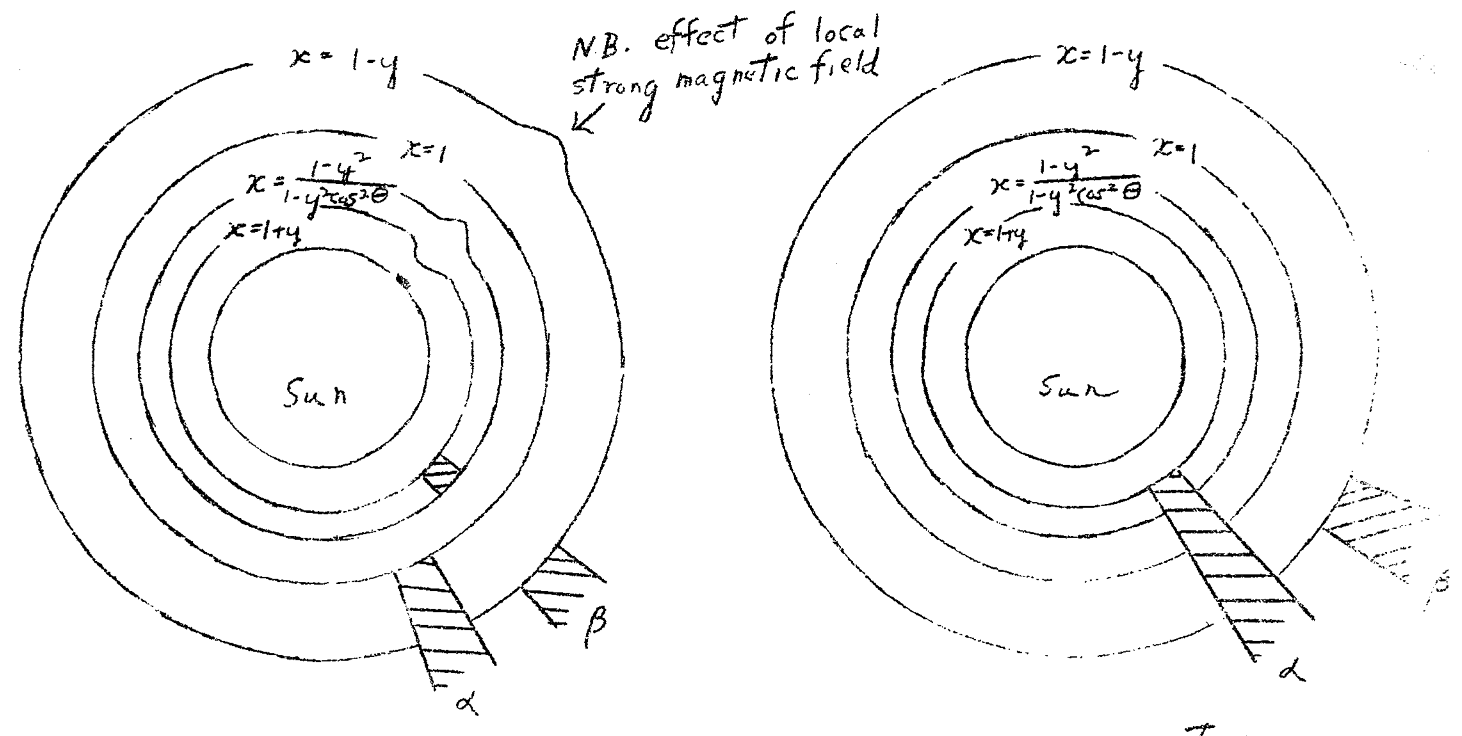

Figure 8. Regions of uninhibited propagation in the solar atmosphere in a general magnetic field.

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magnetic field, the delineation of these regions (Figures 7 and 8) enables certain qualitative conclusions to be drawn. In general, propagation will be QT so that the regions accessible to observation will be within the zones $0 \leqslant x \leqslant 1$ for wave α and $0 \leqslant x \leqslant 1 - y$ for wave β; the region near the level $x = 1 + y$ is accessible only in the QL directions very close to those of $\pm \underset{\sim}{B}_o$. A fuller discussion for the solar atmosphere is given in Westfold (1950).

2.4 Plasma Oscillations

In §2.2(10) we saw that for an electromagnetic field in an ionized medium with no imposed magnetic field

$$K \operatorname{div} \underset{\sim}{E} = 0 \quad . \tag{1}$$

We took the case where $K \neq 0$, so that the free-charge density

$$\rho_e = \epsilon_v \operatorname{div} \underset{\sim}{E} \tag{2}$$

was zero, and found that a propagated electromagnetic field resulted, in which plane waves proceeded according to the complex refractive index q where $q^2 = K$.

Let us now consider the other alternative offered by the equation $\rho_e K = 0$, viz.

$$\rho_e \neq 0, \ K = 0. \tag{3}$$

Then, since

$$\rho_e = \rho_e' \, e^{-i\omega t}, \ \underset{\sim}{E} = \underset{\sim}{E}' \, e^{-i\omega t}, \text{ etc.,} \tag{4}$$

we conclude that at any point ρ_e and the field vectors oscillate with frequency and damping determined by the condition (3). We say that the medium is executing plasma oscillations. Thus ω is a complex quantity determined by ω_o^2 and ν. By Sec 2.2(7) we have

$$x = 1 + iz,$$

i.e.,

$$\omega^2 + i\omega\nu - \omega_o^2 = 0,$$

giving

$$\omega = -(1/2)i\nu \pm \sqrt{(\omega_o^2 - (1/4)\nu^2)} \quad . \tag{5}$$

In contrast to the case of wave propagation the spatial dependence of ρ_e, $\underset{\sim}{E}$ and $\underset{\sim}{B}$ is independent of ω. Instead of Sec 2.2(13) we have

$$\text{curl curl } \underset{\sim}{E} = 0 \quad . \tag{6}$$

The simplest case is where the field depends only on the Cartesian component $\zeta = \underset{\sim}{n} \cdot \underset{\sim}{r}$. Then (6) gives

$$\frac{d^2 E_\perp}{d\zeta^2} = 0$$

where $\underset{\sim}{E}_\perp = \underset{\sim}{n} \wedge (\underset{\sim}{E} \wedge \underset{\sim}{n})$ is the component of $\underset{\sim}{E}$ perpendicular to $\underset{\sim}{n}$. Thus we get

$$\underset{\sim}{E}_\perp = (\underset{\sim}{A}_1 \zeta + \underset{\sim}{A}_2) e^{-i\omega t} \quad , \tag{7}$$

where $\underset{\sim}{A}_1 \cdot \underset{\sim}{n} = \underset{\sim}{A}_2 \cdot \underset{\sim}{n} = 0$. The field equations impose no restriction on the ζ-dependence of the component $\underset{\sim}{E}_\parallel = \underset{\sim}{E} \cdot \underset{\sim}{n}\,\underset{\sim}{n}$ and hence of

$$\rho_e = \epsilon_v \frac{dE_\xi}{d\xi} \, . \tag{8}$$

By §2.2(9)

$$\underset{\sim}{B} = \frac{1}{i\omega} \, \underset{\sim}{n} \wedge \underset{\sim}{A}_1 \, e^{-i\omega t} \, , \tag{9}$$

which is transverse to $\underset{\sim}{n}$. Thus, the transverse oscillatory electromagnetic field in the case of plasma oscillations may be regarded as the limiting case of a field propagated with infinite phase velocity. The longitudinal electric field is associated with the free-charge, or plasma oscillations. The natural frequency and damping constant of the medium are $\sqrt{(\omega_o^2 - (1/4)\nu^2)}$ and $(1/2)\nu$, which reduce to ω_o and 0 when collisional effects are negligible. Hence ω_o is properly called the angular frequency of electron plasma oscillations. The results obtained here are typical of those resulting from the more general decomposition of $\underset{\sim}{E}$ into solenoidal and lamellar components (see Field, 1956).

In the presence of an imposed magnetic field the condition corresponding to (1) is §2.3(12)

$$\text{div } (\underset{\approx}{K} \cdot \underset{\sim}{E}) = 0 \, , \tag{10}$$

which cannot immediately be interpreted in terms of ρ_e. This is possible only in the cases where $\underset{\sim}{E}$ is parallel to one of the principal directions of the complex dielectric tensor $\underset{\approx}{K}$. In terms of the principal components of $\underset{\sim}{E}$ it can be shown that (2) and (10) become

$$\rho_e = \epsilon_v\left(\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3}\right)$$

and

$$K_1\frac{\partial E_1}{\partial x_1} + K_2\frac{\partial E_2}{\partial x_2} + K_3\frac{\partial E_3}{\partial x_3} = 0 ,$$

the medium being uniform. It can now be seen that if the principal components other than E_1 are zero (2) and (10) give

$$\rho_e K_1 = 0 , \qquad (11)$$

and the medium will execute plasma oscillations if $K_1 = 0$. The frequencies and damping constants of the three types of oscillation are determined by the real and imaginary parts of ω as given below:

Type I: $K_1 = 0$

$$x = 1 + iz - y ,$$

i.e.,

$$\omega^2 + \omega(i\nu - \omega_B) - \omega_o^2 = 0,$$

giving

$$\omega = -(1/2)(i\nu - \omega_B) \pm \sqrt{\left\{\omega_o^2 + (1/4)(i\nu - \omega_B)^2\right\}}, \qquad (12)$$

Type II: $K_2 = 0$

$$x = 1 + iz + y ,$$

i.e.,

$$\omega^2 + \omega(i\nu + \omega_B) - \omega_o^2 = 0 ,$$

giving

$$\omega = -(1/2)(i\nu + \omega_B) \pm \sqrt{\{\omega_o^2 + (1/4)(i\nu + \omega_B)^2\}} . \quad (13)$$

Type III: $K_3 = 0$

$$x = 1 + iz ,$$

i.e.,

$$\omega^2 + i\omega\nu - \omega_o^2 = 0 ,$$

giving

$$\omega = -(1/2)i\nu \pm \sqrt{(\omega_o^2 - (1/4)\nu^2)} . \quad (14)$$

The first two types of field, corresponding to the complex principal components E_1 and E_2, represent real electric vectors rotating about the direction of $\underset{\sim}{B}_o$ with angular velocities and damping determined by ω_1, as given by (12), and $-\omega_2$, as given by (13); the third type represents a real electric vector along the direction of $\underset{\sim}{B}_o$ executing damped linear oscillations determined by ω_3, as given by (14). By Sec.2.3(11), Sec.2.2(9), and (2), we have:

Type I: $E_2 = E_3 = 0$

$$\left.\begin{aligned} &\frac{\partial^2 E_1}{\partial x_1 \partial x_2} + \frac{\partial^2 E_1}{\partial x_3^2} = 0 , \\ &B_1 = -\frac{1}{\omega}\frac{\partial E_1}{\partial x_3}, \quad B_2 = 0 , \quad B_3 = \frac{1}{\omega}\frac{\partial E_1}{\partial x_1} , \\ &\rho_e = \epsilon_\nu \frac{\partial E_1}{\partial x_1} . \end{aligned}\right\} \quad (15)$$

<u>Type II: $E_3 = E_1 = 0$</u>

$$\left.\begin{aligned} &\frac{\partial^2 E_2}{\partial x_1 \partial x_2} + \frac{\partial^2 E_2}{\partial {x_3}^2} = 0 , \\ &B_1 = 0, \; B_2 = \frac{1}{\omega}\frac{\partial E_2}{\partial x_3} , \; B_3 = -\frac{1}{\omega}\frac{\partial E_2}{\partial x_2} , \\ &\rho_e = \epsilon_v \frac{\partial E_2}{\partial x_2} . \end{aligned}\right\} \qquad (16)$$

<u>Type III: $E_1 = E_2 = 0$</u>

$$\left.\begin{aligned} &\frac{\partial^2 E_3}{\partial x_1 \partial x_2} = 0 , \\ &B_1 = \frac{1}{\omega}\frac{\partial E_3}{\partial x_2} , \; B_2 = -\frac{1}{\omega}\frac{\partial E_3}{\partial x_1} , \; B_3 = 0 , \\ &\rho_e = \epsilon_v \frac{\partial E_3}{\partial x_3} . \end{aligned}\right\} \qquad (17)$$

<u>Application to non-thermal emission from the Sun</u>

Radio-frequency radiation from the Sun is classified by Pawsey and Bracewell (1955) under the following headings:

(i) Thermal component

(ii) Slowly-varying component

(iii) Noise storms (enhanced radiation)

(iv) Outbursts

(v) Isolated bursts .

We discuss these briefly in turn:

(i) Depends on the kinetic-temperature distribution in the solar atmosphere and the optical depth of various points of the ray trajectories. It is due to "free-free transitions" in encounters between electrons and protons and provides a background below which the brightness cannot fall. We shall consider the thermal component in Chapter 3, with the aid of the knowledge we have gained of ray trajectories and the absorption coefficient.

(ii) Occurs on decimeter wave-lengths. Apparent temperatures reach twice the background value. The variations above the background take place in times of the order of a month, showing a marked 27-day component. Measurements indicate a slight degree of circular polarization. The radiation appears to come from bright areas over sunspot and other active regions. Its origin is possibly thermal arising in local regions of the corona of high density and temperature.

(iii) Occur on meter wave-lengths. Apparent temperatures are of the order of 10^{10}°K. Allen used the term "noise storm" because of the similarity of the records to those of magnetic storms. The phenomenon is characterized by a long series of short-lived bursts, termed Type I bursts by Wild (1951), superimposed on a high enhanced level. Their duration may be anything from hours to days. Noise storms exhibit definite circular polarization and have been associated with areas above sunspots.

(iv) Occur on meter wave-lengths. They are termed Type II bursts by Wild (1950a). Apparent temperatures are of the order of 10^{10}°K. Their duration is for minutes. Type II bursts have been associated with some flares. They exhibit frequency drifts of the order of 1/4 Mc/s per second, which corresponds to speeds through the f_o-levels of the corona of the order of 500 km/sec, of the same order as for the corpuscular streams that are believed to accompany solar flares and to be responsible for the aurorae and for non-recurrent magnetic storms.

(v) Occur on meter wavelengths. They are termed Type III bursts by Wild (1950b). Apparent temperatures are of the order of 10^9°K. Their duration is for seconds. They exhibit frequency drifts of the order of 20 Mc/s per second, which corresponds to very large velocities of the order of 10^5 km/sec. No corresponding physical velocities had been associated with such drifts until Wild, Roberts and Murray (1954) pointed out that the time delays between the onset of solar flares and the increase of terrestrial cosmic-ray intensity sometimes observed was consistent with such particle velocities.

The mechanism of radiation of Types (iii), (iv) and (v) is not known. Since the apparent temperatures quoted can be increased by a factor of about 100 when converted to brightness temperatures over the region of origin, there can be no possibility of assigning such radiation to thermal processes.

Shklovsky (1946) and Martyn (1947) independently suggested that the radiation might be due to macroscopic plasma oscillations of the coronal medium. Such oscillations will have characteristics of frequency and damping such as those indicated above, so that free oscillations cannot be maintained. Martyn therefore suggested a mechanism of excitation due to the breakdown of electric polarization due to turbulent motion in a magnetic field, but no investigation along these lines has been carried through. A difficulty encountered is that of the escape of radiation, since $q = 0$ in the oscillating medium. This difficulty would seem to disappear, as with a magnetron oscillator, if there are boundaries on the other side of which $q \neq 0$ for the frequency of the plasma oscillations. How far these considerations apply to a medium of continually varying refractive index is uncertain.

Jaeger and Westfold (1949) obtained transient solutions following the application of initial values of the electric field, its time derivative and the current and showed that the spectral components of the resulting field were radiated away with the propagation characteristics given by Lorentz theory, the spectral components of greatest amplitude being those close to the plasma frequency. They applied these results to bursts with some success, but left open the question of how these initial conditions might arise. This problem is the subject of current research (Westfold, 1957).

Another non-thermal mechanism which may have application to solar phenomena is the synchrotron mechanism, which, however, would be expected to yield polarized radiation. We shall discuss this mechanism in Chapter 5, particularly as one of the sources of cosmic radiation.

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R A D I O   A S T R O N O M Y

Chapter 3

Thermal Radiation

CHAPTER 3

Thermal Radiation

3.1 The Equation of Transfer

Consider an inhomogeneous isotropic medium in which radiation originates and is propagated. It is simplest to consider the propagated radiation in terms of rays of varying intensity along trajectories determined by the refractive-index structure of the medium and the equations of the electromagnetic field. In general, along each element of a trajectory the ray gains by emission from the medium and loses by absorption. There may also be a loss from incoherent scattering, which is offset by a gain from the scattering of the other rays at that point of the field. The effects of incoherent scattering appear to be negligible in an ionized gas, but there is another type of scattering, coherent scattering, which is responsible for the refractive index of the medium being different from unity. We have already seen that this is of fundamental importance in the propagation of radio-frequency radiation.

We consider a bundle of ray trajectories within the frequency band $(f, f + df)$ that passes normally through the surface element dS into the solid angle $d\Omega$. If I_f is the monochromatic intensity in this direction, we have from our definition of Sec. 1.2 that

$$I_f \, df \, dS \, d\Omega \, dt$$

is the amount of f-radiation flowing through dS into $d\Omega$ in time dt. After traversing a distance ds along the trajectory the intensity will be reduced by the factor $e^{-\kappa_f ds}$ where κ_f is the absorption coefficient, i.e., an amount of radiation

$$\kappa_f \, ds \, I_f \, df \, dS \, d\Omega \, dt$$

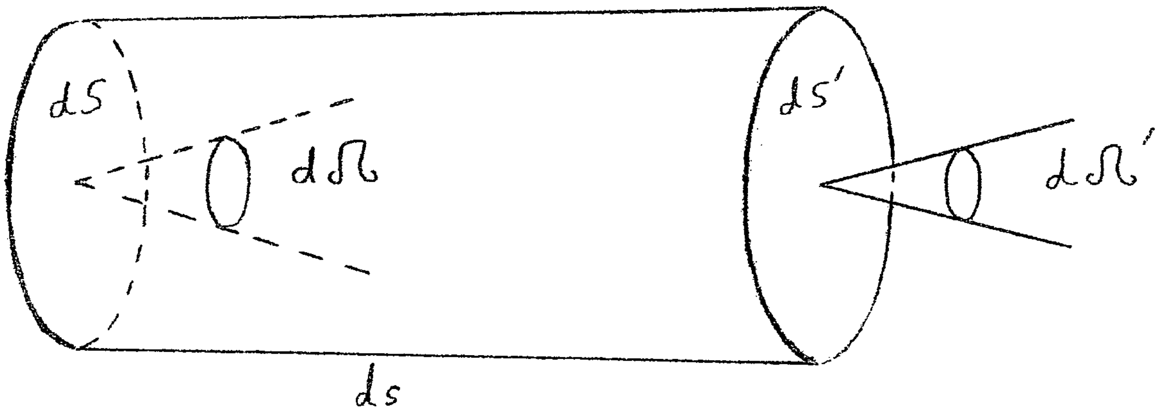

Figure 1. The change in a bundle of ray trajectories in a medium

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is lost by absorption. At this point the original bundle of trajectories now passes normally through the durface dS' into the solid angle $d\Omega'$ and the monochromatic intensity of the radiation has become I_f'. The gain from emission within the volume enclosed between dS and dS' is specified by the monochromatic emissivity, η_f, which is such that the amount of radiation

$$\eta_f \; df \; dV \; d\Omega \; dt$$

is emitted by the volume element dV into the solid angle $d\Omega$ in time dt. In the present case $dV = dS \; ds$ and, since radiant energy is conserved, we have

$$I_f' \; df \; dS' \; d\Omega' \; dt = I_f \; df \; dS \; d\Omega \; dt \; (1 - \kappa_f \; ds) + \eta_f \; df \; dS \; ds \; d\Omega \; dt.$$

Now, by a theorem on ray trajectories in an isotropic medium of varying refractive index μ_f,

$$\mu_f^2 \; dS \; d\Omega = \mu_f'^2 \; dS' \; d\Omega' ,$$

whence

$$\frac{\mu_f^2}{\mu_f'^2} I_f' = I_f \; (1 - \kappa_f \; ds) + \eta_f \; ds.$$

Since

$$\frac{I_f'}{\mu_f'^2} - \frac{I_f}{\mu_f^2} = \frac{d}{ds} \left(\frac{I_f}{\mu_f^2}\right) ds,$$

we have the equdion of transfer along a trajectory

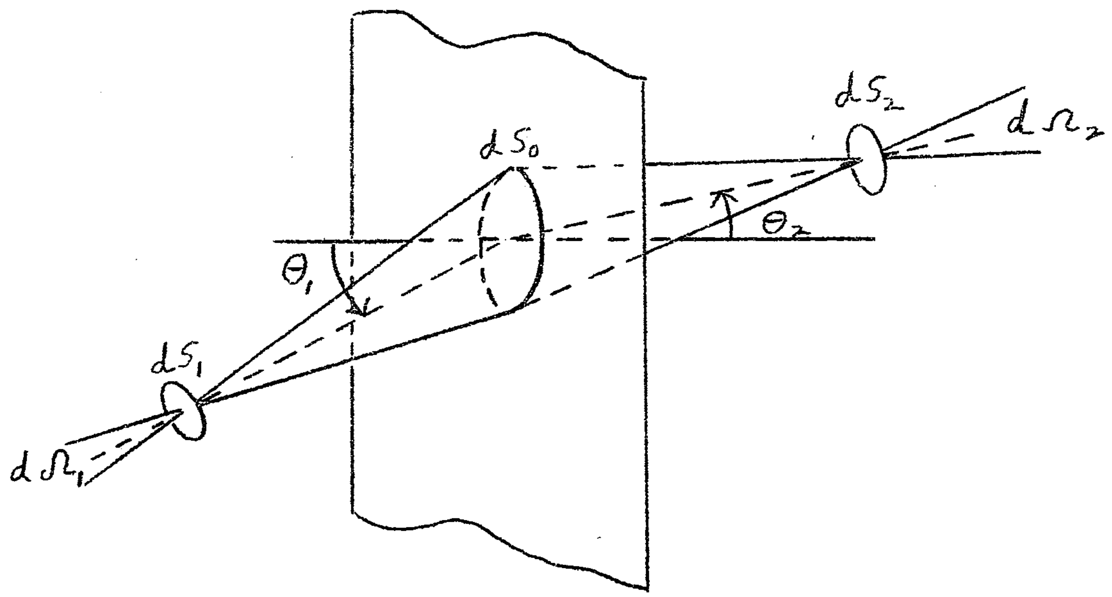

Figure 2. The passage of a bundle of ray trajectories across the interface between two media

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$$\frac{d}{ds}\left(\frac{I_f}{\mu_f^2}\right) = \frac{\eta_f}{\mu_f^2} - K_f \frac{I_f}{\mu_f^2} . \tag{1}$$

This was first obtained for a medium of varying refractive index by Woolley (1947).

Theorem. We proceed to prove the theorem on ray trajectories in an isotropic medium, utilized above. Consider the bundle of ray trajectories that passes normally through dS_1 into the solid angle $d\Omega_1$ in the medium of refractive index μ_1, and intersects the interface with a medium of refractive index μ_2 in the element dS_0 at an angle of incidence θ_1. The bundle is refracted and leaves dS_0 at an angle θ_2 passing normally through dS_2 into the solid angle $d\Omega_2$. If r_1, r_2 are the distances of dS_1, dS_2 from dS_0, we have

$$d\Omega_1 = \frac{dS_0 \cos\theta_1}{r_1^2} , \quad d\Omega_2 = \frac{dS_0 \cos\theta_2}{r_2^2}$$

and

$$dS_1 = r_1^2 \sin\theta_1 \, d\theta_1 \, d\phi , \quad dS_2 = r_2^2 \sin\theta_2 \, d\theta_2 \, d\phi ,$$

where ϕ is the azimuth, specifying the plane of the central trajectory in each medium, which according to Snell's law also contains the normal to dS_0. We also have

$$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2$$

so that

$$\mu_1 \cos\theta_1 \, d\theta_1 = \mu_2 \cos\theta_2 \, d\theta_2 \ .$$

Thus

$$\begin{aligned} dS_1 \, d\Omega_1 &= dS_0 \cos\theta_1 \sin\theta_1 \, d\theta_1 \, d\phi \\ &= dS_0 \frac{\mu_2^2}{\mu_1^2} \cos\theta_2 \sin\theta_2 \, d\theta_2 \, d\phi \\ &= \frac{\mu_2^2}{\mu_1^2} dS_2 \, d\Omega_2 \ . \end{aligned}$$

Hence

$$\mu_1^2 \, dS_1 \, d\Omega_1 = \mu_2^2 \, dS_2 \, d\Omega_2 \ ,$$

or, in an isotropic medium of continuously varying refractive index,

$$\mu^2 \, dS \, d\Omega = \text{constant} \ .$$

Returning to the equation of transfer (1), we see that it is of the first order and linear, requiring an integrating factor $e^{\int K_f \, ds}$. Accordingly, we introduce the optical depth

$$\tau_f(s) = \int_s^\infty K_f(s') \, ds' \ , \tag{2}$$

backward along ray

measured back from emergence where s is effectively infinite. Thus the maximum optical depth $\tau_f(m)$ is that of the whole trajectory $\tau_f(0)$. Since $d\tau_f = -K_f \, ds$, equation (1) becomes $d\tau = -K_f \, ds$

$$\frac{d}{d\tau_f}\left(\frac{I_f}{\mu_f^2}\right) - \frac{I_f}{\mu_f^2} = -\frac{\eta_f}{\mu_f^2 K_f} \ , \tag{3}$$

i.e.,

$$\frac{d}{d\tau_f}\left(\frac{I_f}{\mu_f^2}e^{-\tau_f}\right) = -\frac{\eta_f}{\mu_f^2\kappa_f}e^{-\tau_f}.$$

Integrating from the beginning of the trajectory,

$$\frac{I_f}{\mu_f^2}e^{-\tau_f} = \left(\frac{I_f}{\mu_f^2}e^{-\tau_f}\right)^{(m)} + \int_{\tau_f}^{\tau_f(m)} \frac{\eta_f}{\mu_f^2\kappa_f}e^{-\tau_f}d\tau_f\ , \qquad (4)$$

which gives the intensity at optical depth τ_f in terms of that at the beginning of the trajectory $\tau_f(m)$ and the values of the Ergiebigkeit

$$J_f = \frac{\eta_f}{\mu_f^2\kappa_f} \qquad (5)$$

along the trajectory. The trajectories can be calculated from Snell's law if μ_f is known as a single-valued function of position (see Sec. 2.2). Since observations are concerned with the emergent intensity, where $\tau_f = 0$ and $\mu_f = 1$, we have from (4)

$$I_f = \left(\frac{I_f}{\mu_f^2}e^{-\tau_f}\right)^{(m)} + \int_0^{\tau_f(m)} J_f\, e^{-\tau_f}\, d\tau_f \qquad (6)$$

for the emergent intensity. In an ionized gas μ_f and κ_f are given by the formulae of Sec. 2.2.

3.2 Thermal Radiation from Interstellar Space and the Solar Atmosphere

In interstellar space and the solar atmosphere we suppose that collisions between the ionized gas particles are sufficient for the gas particles to have reached a Maxwellian distribution of velocities. There is then local thermodynamic equilibrium, so that μ_f and K_f are related by Kirchhoff's formula

$$\eta_f = \mu_f^2 \ K_f \ B_f \ , \tag{1}$$

where B_f is the intensity (both polarizations) of black-body radiation of temperature T. Thus, by comparison with Sec. 3.1 (5),

$$J_f = B_f \ , \tag{2}$$

so that J_f depends only on the temperature T of the gas. It is then natural to express Sec. 3.1 (6) in terms of the brightness temperature T_b of the emergent radiation, which will be unpolarized. Thus, since

$$I_f = \frac{2k}{c^2} f^2 \ T_b \ , \quad B_f = \frac{2k}{c^2} f^2 \ T \ , \tag{3}$$

$$T_b = \left(\frac{T_b}{\mu_f^2} e^{-\tau_f}\right)^{(m)} + \int_0^{\tau_f^{(m)}} T \ e^{-\tau_f} \ d\tau_f \ . \tag{4}$$

Now, if we consider the brightness temperature of rays from the galactic interstellar gas, we have rectilinear trajectories since the gas is so tenuous that $\mu_f = 1$. The equation of transfer is therefore the same as for optical radiation. Further, since no galactic radiation contributes to the ray at the beginning of the trajectory $T_b^{(m)} = 0$ and (4) reduces to

$$T_b = \int_0^{\tau_f^{(m)}} T \ e^{-\tau_f} \ d\tau_f \ , \tag{5}$$

where $\tau_f^{(m)}$ is the optical depth of the particular trajectory considered.

If T is constant this becomes

$$T_b = T(1 - e^{-\tau_f^{(m)}}) , \qquad (6)$$

or, in directions where $\tau_f^{(m)} \ll 1$,

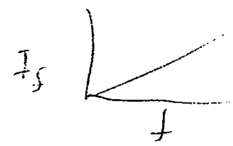
$$T_b = T\, \tau_f^{(m)}. \qquad (7)$$

There is an immediate conclusion to be drawn on the spectrum of thermal radiation from any direction (Piddington, 1951). Since $\kappa_f \propto 1/f^2$ (see Sec. 3.3), (5) may be written

$$T_b(f) = \int_0^{\tau_1^{(m)}} T\, e^{-\tau_1/f^2}\, d\tau_1/f^2 ,$$

whence it is immediately obvious that

$$\frac{d}{df}\,(f^2\, T_b) > 0.$$



In the solar atmosphere we have seen that for any frequency f the surfaces of constant electron density n are the surfaces of constant μ_f (in the absence of an imposed magnetic field). In particular, there is a surface $f_0 = f$ (where $\mu_f = 0$), inside which propagation is inhibited. In the case of spherical symmetry only the radial trajectories reach this level. All the others are bent back, the turning points being at higher levels the farther the trajectories are from the center of the optical disk (see Chapter 2, Figure 3). Again, no solar radiation contributes to the ray at the beginning of the trajectory so that the brightness temperature is given by (5), where the integral is taken over the curved trajectory. With spherical symmetry the temperatures at optical depths τ_f and $2\tau_f^{(0)} - \tau_f$ are the same, where $\tau_f^{(0)} = \frac{1}{2}\tau_f^{(m)}$ is the optical depth of the turning point. Hence we obtain the formula

$$T_b = 2e^{-\tau_f^{(0)}} \int_0^{\tau_f^{(0)}} T \cosh (\tau_f^{(0)} - \tau_f)\, d\tau_f , \tag{8}$$

which is applicable to rays of meter wavelengths for which $\tau_f^{(m)} = 2\tau_f^{(0)}$ is not too large, the trajectories being confined to the corona. For centimeter and decimeter wavelengths, which penetrate into the chromosphere, the absorption may become so great that τ_f is effectively infinite there. In such cases (4) gives

$$T_b = \int_0^{\infty} T\, e^{-\tau_f}\, d\tau_f , \tag{9}$$

as for optical radiation from the photosphere. Smerd (1950) used these results to investigate the brightness distribution of radiation of different frequencies over the solar disk, assuming spherically symmetrical model distributions of electron density and temperature. In particular, for a uniform-temperature corona, (8) gives

$$T_b = T\,(1 - e^{-2\tau^{(0)}}) . \tag{10}$$

This formula was deduced by Burkhardt and Schlüter (1949) from thermodynamic arguments, and applied to coronal rays whose trajectories were defined by Snell's law.

A remarkable conclusion that can be drawn from these formulae is that one should expect limb-brightening at those frequencies for which the trajectories from the central parts of the disk can penetrate into the chromosphere. This prediction was first made by Martyn (1948). The first reliable model distributions of brightness temperature across the disk, exhibiting this feature, were obtained by Smerd.

If we apply (9) to trajectories issuing from infinite optical depth in the chromosphere at uniform temperature T_{ch} and passing out through the corona at uniform temperature T_c, we get a brightness temperature consisting of separate contributions from chromosphere and corona

$$T_b = T_{ch}\, e^{-\tau_c} + T_c\, (1 - e^{-\tau_c}) \ , \tag{11}$$

where τ_c is the optical thickness of the coronal part of the trajectory. We may take $T_{ch} = 10^4$ °K, $T_c = 10^6$ °K, whence for decimeter and centimeter wavelengths τ_c is small. The chief contribution to T_b is then T_{ch}. Away from the center, the trajectories at any frequency eventually no longer penetrate to the chromosphere, so that the only contribution is from the corona, as given by (10). The physical length, and with it the optical thickness, of the coronal portion increases as the trajectory comes from regions farther from the center of the disk, so that the coronal contribution increases to a maximum outside the optical disk. The effect becomes less marked at centimeter wavelengths at which the corona is always optically very thin.

Experimental confirmation of limb brightening has been difficult to obtain. Stanier (1950) was unable to observe it at 500 Mc/s with his two-element interferometer. Smerd's predicted curves were based on spherical symmetry and we recall that Stanier's analysis deduced the brightness across the disk from a strip scan, again assuming symmetry. However, observations scanning the disk in different directions have since been made on the same frequency by Swarup and Parthasarathy (1955) at Sydney, using a multiple-element interferometer, and by O'Brien and Tandberg-Hanssen (1955) at Cambridge using Stanier's equipment. These gave maximum limb-brightening along the equatorial diameter, falling off to limb-darkening along the polar diameter. Observations by Christiansen and Warburton (1955) at 1420 Mc/s in different directions showed similar features. It is not certain whether the Sun was radiating according to a different pattern, owing to the different phase in the sunspot cycle at which these observations were made, or whether Stanier's analysis is in error. However, Smerd and Wild (1957) have shown that Stanier's null result could be ascribed to smoothing due to the finite width of the interferometer fringes.

3.3 Microscopic Thermal Processes in an Ionized Gas

In a uniform ionized gas the fundamental radiative processes that take place are the collisions between the electrons and the positive ions of the medium -- free-free transitions. For radio frequencies $hf \ll kT$ and classical theory suffices.

As is usual in kinetic theory we consider only binary encounters. In an encounter between two charged particles both are accelerated and hence emit radiation which, when incident on the surrounding particles in the medium, induces an electric current which may be regarded as the source of the electromagnetic field in the medium produced by the encounter. The relation between the electric vector of the field and the current is the generalized Ohm's law of Lorentz theory, Sec. 2.2 (1). This process may otherwise be regarded as the coherent scattering of the primary radiation by the surrounding particles, which is responsible for the refractive index of the medium (see Darwin, 1925; Hartree, 1928, 1931). At the same time encounters between the surrounding particles scattering the radiation reduce the induced current. On the average we may take it that the effect of a collision is to extinguish the momentum of the particles relative to the gas as a whole; this is represented by the collisional term $\nu \underset{\sim}{j}$ in the Ohm's-law equation, and is responsible for the refractive index q_f being complex, and hence for the absorption coefficient k_f.

The Cross Section for Absorption. We consider the absorption of radiation by an average electron. The equation of motion for an average individual conduction electron is, according to the above considerations,

$$m\,(\dot{\underset{\sim}{v}} + \nu \underset{\sim}{v}) = e\,\underset{\sim}{E}\,. \tag{1}$$

For f-radiation,

$$\underset{\sim}{v}\,(-i\omega + \nu) = \frac{e}{m}\,\underset{\sim}{E}\,,$$

whence

$$\underset{\sim}{v} = \frac{ie/m\,\omega}{1 + i\nu/\omega}\,\underset{\sim}{E}\,. \tag{2}$$

The rate of absorption of f-radiation is equal to the rate at which the field does work on the electron, viz.,

$$W = \mathrm{Rl}\,\underset{\sim}{v} \cdot \mathrm{Rl}\,e\,\underset{\sim}{E}\,.$$

Thus the average rate of absorption

$$\begin{aligned}\overline{W} &= \frac{1}{2}\,\mathrm{Rl}\left(e\,\underset{\sim}{v} \cdot \underset{\sim}{E}^*\right)\\ &= \frac{1}{2}\,\mathrm{Rl}\left(\frac{ie^2/m\,\omega}{1 + i\nu/\omega}\left|E\right|^2\right),\end{aligned}$$

i.e.,

$$\overline{W} = \frac{1}{2}\,\frac{\nu e^2/m\omega^2}{1 + \nu^2/\omega^2}\left|E\right|^2\,. \tag{3}$$

Now the average Poynting vector (Sec. 2.2 (23)) for a monochromatic plane-wave field travelling in the direction $\underset{\sim}{n}$ is

$$\overline{\underset{\sim}{S}} = \frac{1}{2}\,\epsilon_v\,\mu_f\,c\left|E\right|^2\underset{\sim}{n}\,. \tag{4}$$

Since the cross section for absorption a_f is given by the relation

$$\overline{W} = a_f\,\overline{S}\,, \tag{5}$$

we have

$$a_f = \frac{\nu}{1 + \nu^2/\omega^2} \frac{\omega_0^2/\omega^2}{n\mu_f c} , \tag{6}$$

in terms of the electron plasma frequency ω_0. A similar formula could be obtained for the ion cross section, but this is clearly negligible.

<u>The absorption coefficient.</u> The absorption coefficient is given simply in terms of the absorption cross section

$$\kappa_f = n \, a_f \tag{7}$$

giving

$$\kappa_f = \frac{\omega}{c \, \mu_f} \frac{zx}{1 + z^2} , \tag{8}$$

which agrees with the Lorentz formula Sec. 2.2 (21). If we substitute from Sec. 2.1 (12) for ν we get

$$\kappa_f = \frac{4}{3} n \, n_i \frac{e^4 e_i^{\,2}}{(4\pi\epsilon_v)^3 c m^2 f^2 kT \mu_f} \left(\frac{m}{2\pi kT}\right)^{1/2} \ln \left(1 + \check{\mathcal{V}}_{01}^{\,2}\right) . \tag{9}$$

In c.g.s. e.m.u. $4\pi\epsilon_v = 1/c^2$.

<u>The contribution to the emissivity from an encounter.</u> The emission from a free-free encounter was first calculated by Kramers (1923) in determining a formula for the X-ray absorption coefficient. Here, since the refractive index is unity, the effects of scattering are negligible. Smerd and Westfold (1949) applied Kramers's methods to find the radio-frequency emission from an encounter. They showed that if the Lorentz formula

(8) is valid for the absorption coefficient, then the principle of detailed balancing requires the emission to be modified by the additional factor μ_f. Denisse (1950) came to a similar conclusion. Westfold (1950) showed that in this manner coherent scattering was taken into account.

Kramers's procedure was to make a Fourier analysis of the total radiation emitted in an encounter. Since scattering affects each component differently, we must first make a Fourier analysis of the field vectors in order to calculate the contribution to the emissivity from an encounter. Let us define complex Fourier transforms

$$\bar{\underset{\sim}{E}}(\omega) = \int_{-\infty}^{\infty} \underset{\sim}{E}(t)\, e^{i\omega t}\, dt, \text{ etc.,} \tag{10}$$

so that

$$\underset{\sim}{E}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\underset{\sim}{E}}(\omega)\, e^{-i\omega t}\, d\omega, \text{ etc.} \tag{11}$$

Then the amplitudes of the Fourier components satisfy the same equations as the monochromatic field vectors of the Lorentz theory (Sec. 2.2). Now the principal part of the contribution from an encounter is due to the dipole moment of the accelerating electron.[1] If $\underset{\sim}{a}(t)$ is its acceleration at time t, the transform $\bar{\underset{\sim}{a}}(\omega)$ will correspond to $-\omega^2 \underset{\sim}{\xi}'$ in the formula for the radiation field from a monochromatic dipole of complex moment $e\underset{\sim}{\xi}' e^{-i\omega t}$. Thus[2] the field transforms due to an encounter,

[1]The contribution from the more massive accelerating ion is negligible by comparison.

[2]It is here assumed that the velocities are small compared to the velocity of light, so that relativistic effects are negligible.

taking account of scattering, are

$$\left.\begin{aligned} \underset{\sim}{E} &= \frac{e}{4\pi\epsilon_v c^2 R} (\underset{\sim}{\bar{a}} \wedge \underset{\sim}{\hat{R}}) \wedge \underset{\sim}{\hat{R}} , \\ \underset{\sim}{B} &= q_f \frac{\mu_v e}{4\pi c R} \underset{\sim}{\bar{a}} \wedge \underset{\sim}{\hat{R}} , \end{aligned}\right\} \tag{11}$$

where $\underset{\sim}{R}$ is the position vector of the field point relative to the region of the encounter and q_f the complex refractive index for f-radiation. Now the total contribution from an encounter lasting from $t = -\infty$ to $t = +\infty$ is

$$Q_0 = \int_0^\infty Q_f \, df = \int_{-\infty}^{\infty} dt \int \underset{\sim}{S} \circ \underset{\sim}{n} \, d\sigma . \tag{12}$$

and

$$\int_{-\infty}^{\infty} s\underset{\sim}{S}(t) \, dt = \int_{-\infty}^{\infty} \underset{\sim}{E}(t) \wedge \underset{\sim}{H}(t) \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \underset{\sim}{\bar{E}}(-\omega) \wedge \underset{\sim}{\bar{H}}(\omega) \, d\omega$$

$$= \frac{1}{\pi} \int_0^\infty \mathrm{Rl}\left\{ \bar{E}(-\omega) \wedge \bar{H}(\omega) \right\} d\omega$$

by one of the Parseval formulae. Thus

$$Q_f = 2 \int \mathrm{Rl} \left\{ \underset{\sim}{\bar{E}}(-\omega) \wedge \underset{\sim}{\bar{H}}(\omega) \right\} \circ \underset{\sim}{n} \, dS .$$

By (11)

$$\underset{\sim}{\bar{E}}(-\omega) \wedge \underset{\sim}{\bar{H}}(\omega) = q_f \frac{e^2}{16\pi^2 \epsilon_v c^3 R^2} |\bar{a}|^2 \sin^2 \Theta \;\; \underset{\sim}{\hat{R}}$$

where $a \cos \Theta = \underset{\sim}{a} \circ \underset{\sim}{\hat{R}}$. Thus, since $\int \sin^2 \Theta \, d\Omega = 8\pi/3$,

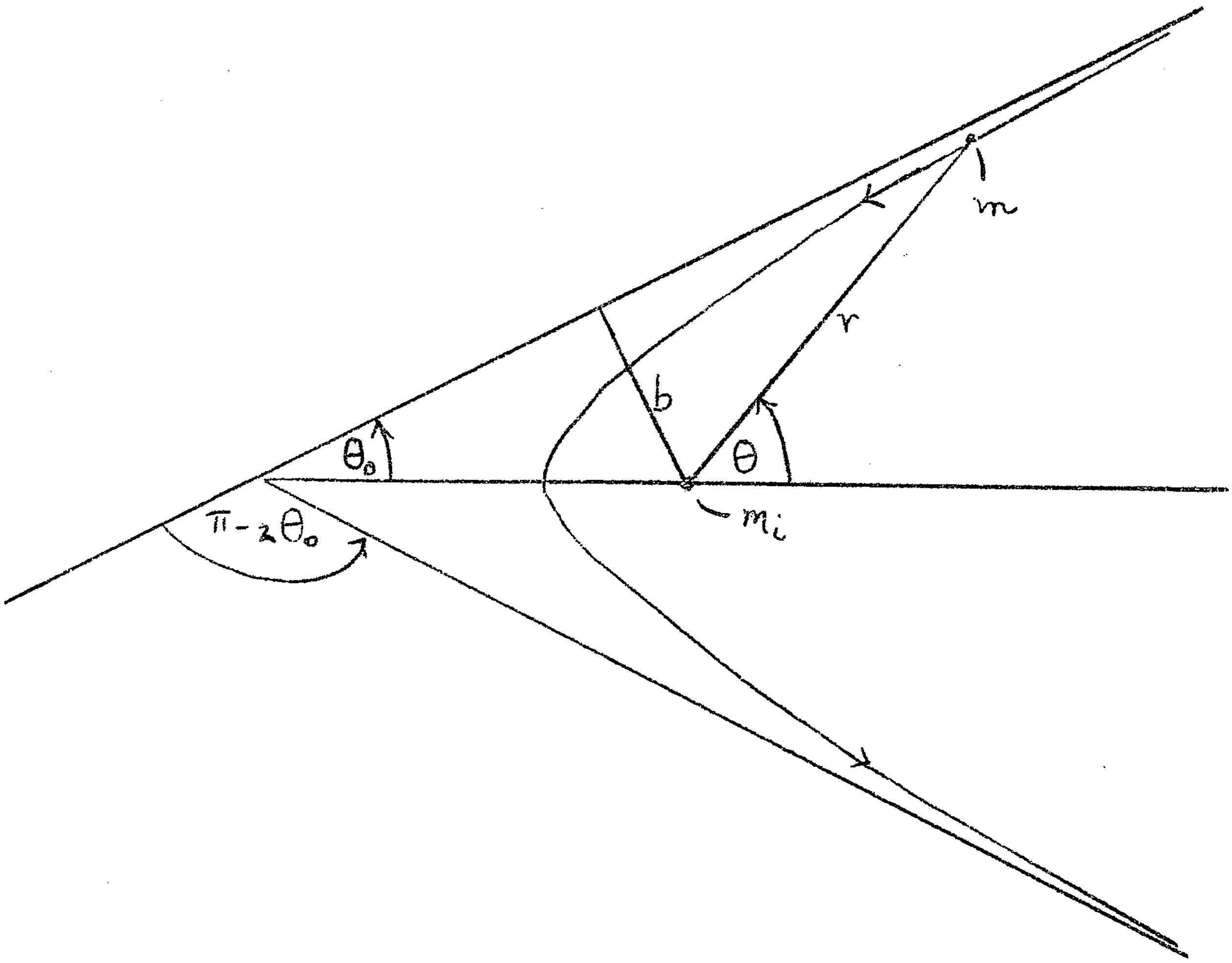


Figure 3. The relative orbit in an encounter between two charges

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$$Q_f = \frac{e^2}{3\pi\epsilon_v c^3}\mu_f|\bar{a}|^2 . \tag{13}$$

Although for a particular enounter the contribution to Q_f depends on the direction, in any sample volume, encounters will be taking place at all orientations so that the average contribution to the emissivity may be taken as $Q_f/4\pi$. We note that the only effect of scattering is to introduce the additional factor μ_f.

<u>The dynamics of an electron-ion encounter.</u> The equation of motion of an electron e, m relative to a heavy ion e_i, m_i is to a good approximation

$$m\ddot{\underset{\sim}{r}} = \frac{e\,e_i}{4\pi\epsilon_v r^3}\underset{\sim}{r} , \tag{14}$$

the familiar inverse-square law, of attraction since e_i is of opposite sign to e. The electron orbit for a free-free encounter is a hyperbola, which can be specified in the plane of the orbit by the relative speed g at infinity and the distance b of m_i from either of the asymptotes, known as the impact parameter.

It is convenient to introduce the dimensionless parameters (Chapman and Cowling, 1952, Sec. 10.3)

$$\upsilon = b/r , \quad \upsilon_0 = -\frac{4\pi\epsilon_v m g^2 b}{e\,e_i} . \tag{15}$$

Then if the axis of the orbit is taken along $\theta = 0$, its equation is easily found to be

$$\nu \nu_0 = 1 \mp \sqrt{(1 + \nu_0^2)} \cos \theta \; .$$

The asymptotes are then along directions given by $\nu = 0$, which is where $\cos \theta = \pm 1/\sqrt{(1 + \nu_0^2)}$. Thus we have the geometrical interpretationsof ν_0

$$\nu_0 = \tan \theta_0 , \; 0 \leqslant \theta_0 \leqslant \pi , \tag{16}$$

where $\theta = \theta_0$ is the initial asymptote. Then the equation of the orbit is

$$\nu \tan \theta_0 = 1 - \cos \theta / \cos \theta_0 \; , \tag{17}$$

so that the final asymptote is $\theta = 2\pi - \theta_0$ and the deflection $\pi - 2\theta_0$. For encounters between unlike charges the possible range of θ_0 is from 0 to $\pi/2$, of ν_0 from 0 to ∞, and of the deflection from π to 0.

The location of e on the orbit at time t is given by the angular-momentum integral

$$r^2 \dot{\theta} = g\, b \; . \tag{18}$$

In terms of the dimensionless time parameter

$$\tau = gt/b \tag{19}$$

and ν we have

$$\frac{d\theta}{d\tau} = \nu^2 , \tag{20}$$

which gives on integration

$$\tau = - \frac{(1 + u_0^2) u}{u^2 - u_0^2} + \frac{1 - u_0^2}{2u_0} \ln \frac{u + u_0}{u - u_0} , \tag{21}$$

where

$$u = \tan(\theta/2) \text{ and } u_0 = \tan(\theta_0/2).$$

In order to determine $|\bar{a}|^2$ in (13) we require $\underset{\sim}{a}(t)$. Resolving (14) into directions parallel and perpendicular to the axis $\theta = 0$ we have, by (15),

$$\frac{b}{g^2} a_{\parallel} = -\frac{\upsilon^2}{\upsilon_0} \cos\theta , \quad \frac{b}{g^2} a_{\perp} = -\frac{\upsilon^2}{\upsilon_0} \sin\theta ,$$

whence, by (20),

$$\frac{b\upsilon_0}{g^2} a_{\parallel} = -\cos\theta \frac{d\theta}{d\tau} , \quad \frac{b\upsilon_0}{g^2} a_{\perp} = -\sin\theta \frac{d\theta}{d\tau} . \tag{22}$$

The transforms are now given by (10) which can be represented by integrals with respect to θ, $\bar{A}_{\parallel}$, $\bar{A}_{\perp}$, where

$$\left.\begin{aligned} \frac{\upsilon_0}{2g} \bar{a}_{\parallel} &= \bar{A}_{\parallel} = -\frac{1}{2}\int_{\theta_0}^{2\pi - \theta_0} e^{i\omega t} \cos\theta \, d\theta , \\ \frac{\upsilon_0}{2g} \bar{a}_{\perp} &= \bar{A}_{\perp} = -\frac{1}{2}\int_{\theta_0}^{2\pi - \theta_0} e^{i\omega t} \sin\theta \, d\theta , \end{aligned}\right\} \tag{23}$$

in which $\omega t = (\omega b/g)\tau$, where τ is given by (21) in terms of θ. The contribution to the emissivity from an encounter specified by the parameters g, υ_0 is thus

$$Q_f = \frac{4e^2}{3\pi\epsilon_v c^3} \mu_f \frac{g^2}{\upsilon_0^2} |\bar{A}|^2 \tag{24}$$

where

$$|\bar{A}|^2 = |\bar{A}_{\parallel}|^2 + |\bar{A}_{\perp}|^2 .$$

<u>The emissivity.</u> The emissivity η_f is now obtained by summing the contributions $Q_f/4\pi$ from all types of encounter that take place in a small volume in a small time interval. Let $\phi(g, \upsilon_0)\, dg\, d\upsilon_0$ be the number of (g, υ_0) encounters per unit volume and time as given by kinetic theory. Then, if υ_{01} is the maximum value of υ_0 for a given value of g,

$$\eta_f = \frac{1}{4\pi}\int_0^{\infty} dg \int_0^{\upsilon_{01}} \phi(g, \upsilon_0)\, Q_f\, d\upsilon_0\, dg. \qquad (25)$$

The reason we have not taken υ_{01} as ∞ is that the inner integral is improper there. (The value $\upsilon_{01} = \infty$ corresponds to an infinite value for b and zero deflection.) There is a significant contribution from small-deflection encounters. Of course, in a gas, the presence of the surrounding molecules effectively prevents binary encounters for υ_0 greater than some (average) limit υ_{01}. For interaction forces represented by an inverse-power law greater than the second the contribution from large-υ_0 encounters is so small that a negligible contribution to (25) comes from (small-deflection) encounters specified by $\upsilon_0 > \upsilon_{01}$ and υ_{01} is conveniently taken as ∞.

Now if there has been sufficient time for collisions to have brought about a Maxwellian distribution of velocities the number of (g,b) encounters per unit volume and time will be

$$8\pi^2 nn_i \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mg^2/2kT}\, b\, db\, g^3\, dg.$$

Substituting from (15) for b while keeping g constant, we get

$$\phi(g, \upsilon_0)\, d\upsilon_0\, dg = 8\pi^2 m n_i \left(\frac{ee_i}{4\pi\epsilon_v m}\right)^2 \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-mg^2/2kT}\ \upsilon_0\, d\upsilon_0 \frac{dg}{g}, \quad (26)$$

whence, by (24),

$$\eta_f = \frac{16}{3}\, n n_i \frac{e^4 e_i^{\,2} \mu_f}{(4\pi\epsilon_v)^3 c^3 m^2} \left(\frac{m}{2\pi kT}\right)^{\frac{1}{2}} G, \quad (27)$$

where

$$G = \frac{m}{kT} \int_0^{\infty} e^{-mg^2/2kT}\, g\, dg \int_0^{\upsilon_{01}} |\bar{A}|^2 \frac{d\upsilon_0}{\upsilon_0}. \quad (28)$$

This formula differs from the corresponding optical formula only by the factor μ_f. The dimensionless factor G was incorrectly evaluated by Kramers (1923) and hence required a correction factor involving numerical as well as quantum considerations when applied to free-bound transitions (Gaunt, 1930; Elwert, 1948, Westfold, 1949).

The evaluation of G. The integrals (23) for $\bar{A}_\parallel$ and $\bar{A}_\perp$ which occur in (28) cannot be evaluated in terms of known functions in the general case. However, suitable approximations for the astrophysical conditions in which we are interested can be obtained by virtue of the fact that the effective duration of a (g,b) encounter, from the point of view of emission of radiation, is specified by the range $-1 \leq \tau \leq 1$, where τ is given by (19).

Figure 4 shows plots of the fraction of the total radiation Q_0 emitted between the instants $\pm\tau$ for a series of values of θ_0. We recall that for electron-positive ion encounters $0 \leq \theta_0 \leq 90°$. The

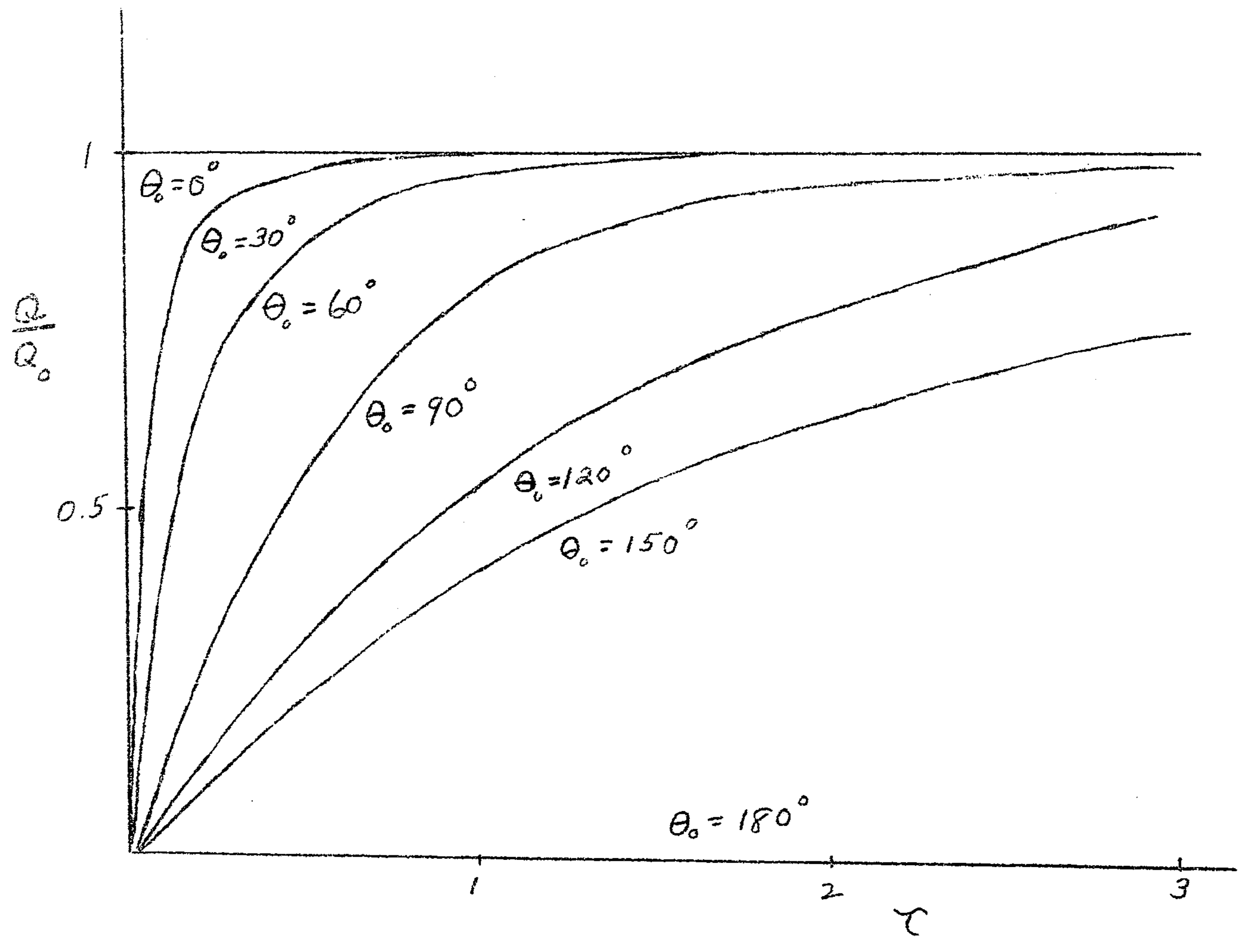

Figure 4. The fraction of the total radiation from an encounter emitted between the instants $\pm\,\tau$

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latter undeviated orbit is the least favorable case, and here 80% of the radiation is emitted between $\tau = \mp 1$, 96% between $\tau = \mp 2$, etc. Thus we may take $-1 \leq \tau \leq 1$ as the duration of an encounter.

The integrals (23) now yield a simple approximation for $\bar{A}_\parallel$, $\bar{A}_\perp$ under the condition $\omega b/g \ll 1$. In this case $\omega t = (\omega b/g)\tau \ll 1$ over the duration of the encounter and we may replace the exponential terms by unity to get on integration

$$\left.\begin{aligned} \bar{A}_\parallel &= \sin\theta_0 = \frac{\upsilon_0}{\sqrt{(1+\upsilon_0^2)}}\,, \\ \bar{A}_\perp &= 0\,. \end{aligned}\right\} \tag{29}$$

To this approximation

$$|\bar{A}|^2 = \frac{\upsilon_0^2}{1+\upsilon_0^2}\,, \tag{30}$$

independent of g and f. This formula may be substituted into (28) if our condition is satisfied for the greatest possible value d of b and a mean value $\overline{1/g}$ for $1/g$. Then, for $\omega d\,(\overline{1/g}) \ll 1$, i.e.,

$$\frac{1}{2}\, nfd \left(\frac{2\pi m}{kT}\right)^{\frac{1}{2}} \ll 1\,,$$

$$G = \frac{m}{2kT}\int_0^\infty e^{-mg^2/2kT}\ \ln\left\{1 + \upsilon_{01}^2(d)\right\} g\, dg\,,$$

where

$$\upsilon_{01}(d) = -\frac{4\pi\,\epsilon_v\, mg^2 d}{ee_i}\,.$$

We take d as the mean distance between pairs of interacting molecules, since for $b > d$ an encounter cannot be properly regarded as binary. The logarithmic term varies only slowly over the range of g so that $\upsilon_{01}(d)$ can be replaced by its mean value $(\overline{mg^2} = 4\ kT)$ allowing the further integration

$$G = \frac{1}{2} \ln \left\{ 1 + \overline{\upsilon_{01}(d)^2} \right\} , \qquad (31)$$

where

$$\overline{\upsilon_{01}(d)} = - \frac{16\ \pi\ \epsilon_v kTd}{ee_i} . \qquad (32)$$

In c.g.s. units (32) gives

$$\overline{\upsilon_{01}(d)} = 2.4 \times 10^3\ dT/Z$$

where $e_i = - Ze$; G is given by (31) provided that

$$fdT^{-1/2} \ll 10^6 .$$

This condition is satisfied at radio frequencies in the solar atmosphere and in the H II clouds of interstellar space, where also $\overline{\upsilon_{01}(d)} \gg 1$ so that

$$G = \ln \overline{\upsilon_{01}(d)} . \qquad (31')$$

If $\omega \overline{d(1/g)} > 1$ we may expect (30) to give a good approximation in (28) for values of υ_0 up to $\upsilon_{01}(g/\omega)$, where $\omega b/g = 1$. Its mean value is

$$\overline{\nu_{01}(g/\omega)} = -\frac{15\,\pi\,\epsilon_v(kT)^{3/2}}{ee_1f(2\pi m)^{1/2}}\,, \tag{33}$$

i.e.,

$$\overline{\nu_{01}(g/\omega)} = 3.5 \times 10^8\, T^{3/2}/Zf$$

in c.g.s. units. In the extended H II regions of interstellar space, where (31) is inapplicable $\overline{\nu_{01}(g/\omega)} > 1$, so that in calculating the contribution to (28) from $\nu_{01}(g/\omega)$ to $\nu_{01}(d)$ we may use the approximate formulae for $\overline{A}_{\parallel}$, $\overline{A}_{\perp}$ valid for large ν_0. The calculation (Westfold, 1951) shows that

$$\int_{\nu_{01}(g/\omega)}^{\nu_{01}(d)} < \int_{\nu_{01}(g/\omega)}^{\infty} |\overline{A}|^2 \frac{d\nu_0}{\nu_0} = 0.25,$$

whereas $\int_0^{\nu_{01}(g/\omega)}$ has a mean value of order 10 in the extended H II regions. Thus for

$$\frac{1}{2}\pi f d \left(\frac{2\pi m}{kT}\right)^{1/2} > 1$$

we take

$$G = \frac{1}{2}\ \ln\left\{1 + \overline{\nu_{01}(g/\omega)^2}\right\}, \tag{34}$$

instead of (31).

The two approximate formulae for G represent slowly varying functions so that their precise form is not of great importance in calculating the emissivity from (27). We may take (31) for

$$\frac{1}{2}\,\pi f d\left(\frac{2\pi m}{kT}\right)^{1/2} < 1 ,$$

suitable to the solar atmosphere and the H II clouds, and (34) for

$$\frac{1}{2}\,\pi f d\left(\frac{2\pi m}{kT}\right)^{1/2} > 1 ,$$

suitable to the extended H II regions. Like Denisse (1950), we might have included a lower limit in the integration corresponding to an atomic dimension, in place of 0. This, however, has an insignificant effect on the value of G.

We should here comment on Cohen, Spitzer and Routly's (1950) (see also Spitzer, 1956) conclusion that the upper limit of b should be the Debye distance

$$\lambda_D = \left\{\epsilon_v kT/(ne^2 + n_i e_i{}^2)\right\}^{1/2} ,$$

beyond which the shielding effect of the surrounding molecules effectively neutralizes the central positive-ion field of attraction. They found that for an electron initially at rest the mean rate of change of the square of the speed, due to statistical fluctuations of the electrostatic field, is correctly given by the binary-encounter formulae even for $b > d$; the appropriate upper limit is taken as λ_D since beyond this distance the macroscopic fields take effect. From the point of view of emission of radiation it is only permissible to take this as the upper limit of b if the accelerating charges are interacting in the same way as if the encounter were binary. It is difficult to see how this can be the case beyond the interionic distance d. If the effects of the surrounding molecules are to be taken into account it would be better to use a Debye potential* instead of a Coulomb potential. Then in (25) it would be possible to integrate between limits corresponding to b = 0 and ∞ without the integral becoming improper.

<u>The absorption coefficient.</u> With a Maxwellian distribution of velocities there is a detailed balancing of microscopic processes and hence thermodynamic equilibrium in the gas. By Sec. 3.2 (1) we then have

$$K_f = \frac{c^2}{2kTf^2\mu_f^2}\eta_f . \qquad (35)$$

Substitution for η_f should then give the same result (9) as was obtained directly from consideration of the collision cross section. From (27) we get

$$K_f = \frac{8}{3} nn_i \frac{e^4 e_i^2}{(4\pi\epsilon_v)^3 cm^2 f^2 kT\mu_f} \left(\frac{m}{2\pi kT}\right)^{1/2} G, \qquad (36)$$

which agrees with (9) since $G = \frac{1}{2} \ln (1 + \overline{\nu_{01}^2})$. In c.g.s. units (36) gives

$$K_f = 9.76 \times 10^{-2} \, (nn_i Z^2/f^2 T^{3/2} \mu_f) \, G \ \ cm^{-1} .$$

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*Provided the electrostatic energy remains small compared with the thermal energy of the electron, the right-hand side of (14) would then be multiplied by the additional factor $(1 + r/\lambda_D) \exp(-r/\lambda_D)$.

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Chapter 4

Line Radiation

CHAPTER 4

Line Radiation

4.1 Introduction

In discussing the differences between radio and optical observations we have already referred to the fact that only a narrow band of frequencies is accepted by the radio instrument, whereas normally a wide band of optical frequencies is accepted by the telescope. Under dispersion, the light from a star or nebula reveals a spectrum containing a number of lines, seen in emission or absorption, from which inferences concerning such things as its composition, state, and radial velocity can be drawn. Historically, the spectral lines were observed long before they were satisfactorily accounted for by theory. By contrast, the techniques of radio observations would render it difficult to discover spectral lines without the aid of theoretical predictions. Indeed, the theoretical predictions indicate only a very few possibilities and of these only the 21-cm line, predicted by van de Hulst (1945) and Shklovsky (1949), has so far been detected. It is due to transitions between hyperfine sub-levels of the ground state of atomic hydrogen and has been observed in galactic H I regions and more recently in the Magellanic Clouds (Kerr, Hindman, and Robinson, 1954) and the external galaxies M 31 (van de Hulst, Raimond, and van Woerden, 1957) M 33 (Raimond and Volders, 1957), and others for which the results have not yet been published.

The first to detect 21-cm radiation were Ewen and Purcell (1951), closely followed by Muller and Oort (1951) in Holland and Christiansen and Hindman (Pawsey, 1951) in Australia.

In the Galaxy the observed Doppler shift and line profiles from different directions have enabled a quite detailed picture of the distribution of rotational velocity and the density distribution in different directions in the galactic plane to be drawn, indicating considerable spiral-arm structure and a total amount of neutral hydrogen equal to about one or two per cent of the total mass of the Galaxy. The resolving power and sensitivity of the equipments used were not sufficient to justify the drawing of very detailed pictures of the Magellanic Clouds and the external galaxies. However, it does appear that the mean radial velocities of the two clouds are different, that there is a systematic change in radial velocity across each, and that neutral hydrogen accounts for about 20 per cent of the total mass; the observations on M 31 have been reconciled with a model showing differential motion of rotation, for which the total mass is three or four times that of the Galaxy and the amount of neutral hydrogen about one per cent of the total mass.

4.2 The absorption coefficient and emissivity

The macroscopic absorption coefficient $K(f)$ for line radiation of frequency f is readily obtained from the atomic cross section for absorption $a(f)$, usually termed the atomic absorption coefficient. Let n_1, n_2

be the number densities of atoms in the lower and higher levels between which the transitions occur and let the corresponding statistical weights be g_1, g_2. Then the ratio of numbers of atoms in the two states is given in terms of an excitation or spin temperature T_s as

$$\frac{n_2}{n_1} = \frac{g_2}{g_1} e^{-hf/kT_s}. \qquad (1)$$

In the galactic H I clouds it is reckoned that collisions are sufficiently frequent to determine the relative populations, in which case the excitation temperature will be identical with the kinetic temperature T of the gas. Now the macroscopic absorption coefficient $K(f)$ represents the net absorptive effect of atoms per unit area of cross section and per unit distance after passage of the radiation, i.e. the net effect of "true" absorption and stimulated emission (or negative absorption). Since the ratio of the Einstein coefficient of stimulated emission B_{21} to the coefficient of absorption B_{12} is given by

$$\frac{B_{21}}{B_{12}} = \frac{g_1}{g_2},$$

the atomic coefficients of "negative" and positive" absorption are in the same ratio so that we have

$$K(f) = (n_1 - \frac{g_1}{g_2} n_2) \; \alpha(f)$$

$$= n_1(1 - e^{-hf/kT_s}) \; \alpha(f),$$

by (1). Since for centimeter wavelengths $hf \ll kT_s$, this reduces to

$$K(f) = n_1 \frac{h\,f}{kT_s} \alpha(f). \qquad (2)$$

Now in terms of the Einstein coefficient B_{12} the atomic absorption coefficient is given by the usual formula

$$\alpha(f) = \frac{hf}{4\pi} B_{12} \phi(f),$$

where $\phi(f)$ df is the probability that the (1,2) transition occurs in the frequency range (f, f + df), and

$$B_{12} = \frac{g_2}{g_1} \frac{c^2}{2hf^3} A_{21},$$

where A_{21} is the Einstein coefficient of spontaneous emission. Hence we have

$$K(f) = n_1 \frac{hf}{kT_s} \frac{c^2 A_{21}}{8\pi f^2} \frac{g_2}{g_1} \phi(f). \qquad (3)$$

Clearly the line profile is specified by the probability $\phi(f)$.

In the present case the natural line profile is masked by the Doppler broadening due to the random velocities of the clouds (and the atoms contained therein) superimposed on the local mass or differential velocity of the gas. We are concerned only with the line-of-sight components V, i.e. with the radial velocities. Let $N(V, \underset{\sim}{r})$ dV be the number density of atoms at the position $\underset{\sim}{r}$, having radial velocities in the range (V, V + dV). Then we must have

$$\int N(V, \underset{\sim}{r})\ dV = n(\underset{\sim}{r}),$$

the local number density, and

$$\int V\ N(V, \underset{\sim}{r})\ dV = n(\underset{\sim}{r})\ V_g\ (\underset{\sim}{r}),$$

where $V_g(\underset{\sim}{r})$ is the radial component of the local differential velocity. If f_0 is the frequency emitted by an atom at rest these atoms will emit radiation in the band $(f, f + df)$ where

$$f = f_0(1 - V/c). \tag{4}$$

It follows that

$$n_1(\underset{\sim}{r})\ \phi(f)\ df = - \frac{g_1}{g_1 + g_2} N(V, \underset{\sim}{r})\ dV,$$

whence, since $df = - (f_0/c)\ dV$, (3) gives

$$K(f) = \frac{g_2}{g_1 + g_2} \frac{hf}{kT_s} \frac{c^2 A_{21}}{8\ \pi f^2} \frac{c}{f_0} N(V, \underset{\sim}{r}) \tag{5}$$

at the position $\underset{\sim}{r}$. Inserting values appropriate to 21-cm radiation, $g_2/g_1 = 3$, $A_{21} = 2.84 \times 10^{-15}$, $f \doteq f_0 = 1.4204 \times 10^9\ sec^{-1}$, we get

$$K(f) = \beta\ N(V, \underset{\sim}{r})/T_s\ , \qquad cm^{-5}\ sec\ \ ^{\circ -1}$$

where $\beta = 5.45 \times 10^{-14}\ cm^3\ sec^{-1}\ {}^{\circ}K$.

In terms of the quantities already defined the volume emissivity $\eta(f)$ is readily seen to be given by

$$\eta(f) = \frac{hf}{4\pi} n_2\ A_{21}\ \phi(f).$$

With $hf \ll kT_s$, (1) gives $n_2 = n_1 g_2/g_1$, whence by (3) we have

$$\eta(f) = \frac{2k}{c^2} f^2 \, T_s \, K(f), \qquad (6)$$

which is Kirchhoff's law for radio-frequency line radiation, where the refractive index is unity. By virtue of this relation, the transfer theory developed in Chapter 3 may be taken over for line radiation; it is simply necessary to substitute the spin temperature T_s for the kinetic temperature T and to use the formula (5) instead of Sec. 3.3 (36) for the absorption coefficient.

4.3 Optical depth and brightness temperature of galactic radiation

As in Chapter 3, the brightness temperature of line radiation, from any direction specified by galactic coordinates (ℓ, b) depends on the distribution of spin temperature and optical depth along the ray trajectory. Up to the present time, analyses of galactic 21-cm radiation have been based on the assumption of a uniform spin temperature T_s for all H I clouds. The assumption is provisional and will, no doubt, be modified when more observational material comes to hand. For the present, therefore, we take the result Sec. 3.2 (6)

$$T_b(f; \ell, b) = T_s(1 - e^{-\tau(f; \ell, b)}), \qquad (1)$$

where

$$\tau(f; \ell, b) = \int_0^\infty K(f)\,dr \tag{2}$$

is the galactic optical thickness of the trajectory (ℓ, b) for radiation of frequency f. The corresponding radial velocity V is given by Sec. 4.2 (4) so that by Sec. 4.2 (5)

$$\tau(f) = \beta \int_0^\infty \frac{N}{T_s}\,dr\ . \tag{3}$$

With a dispersion of cloud velocities everywhere, there will be contributions to the integral in (3) all along the trajectory, but these will come principally from the neighborhood of the points where the differential velocity component V_g is equal to $- c(f - f_0)/f_0$. For 21-cm radiation the value $V_g = 1$ km/sec corresponds to $\Delta f = f - f_0 = - 4.73$ kc/s. The Leiden observations were first reduced on the basis of a dispersion following a law of the form $\exp(-|V - V_g|/\eta)$, as found by Blaauw (1952) in his studies on interstellar absorption lines, but it was found that a better fit was obtained with the assumption of a Gaussian distribution. For the present we shall neglect any dispersion about the local differential velocity V_g and assume that $N(V, \underset{\sim}{r})$ is given by the delta-function* expression

$$N(V, \underset{\sim}{r}) = n(\underset{\sim}{r})\ \delta(V - V_g(\underset{\sim}{r})\ ). \tag{4}$$

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*The delta function $\delta(x)$ is an improper function whose value is zero everywhere except at $x = 0$ where it is infinite to an order such that $\int_{-\infty}^{\infty} \delta(x)\,dx = 1$. From this follows the "sifting" property, $\int f(x)\,\delta(x - a)\,dx = f(a)$, where $x = a$ is within the range of integration.

Substituting from Sec. 4.2 (4), and (4) into (3) we get

$$\tau(f) = \beta \int_0^{\infty} \frac{n}{T_s} \delta(- c\frac{\Delta f}{f_0} - V_g)\, dr.$$

We transform the variable of integration to V_g, so that the sifting property of the delta function may be utilized. Thus

$$\tau(f) = \beta \int_{-\infty}^{\infty} \frac{n}{T_s} \delta(- c\frac{\Delta f}{f_0} - V_g) \frac{dV_g}{dV_g/dr},$$

whence

$$\tau(f) = \beta \sum \frac{n/T_s}{dV_g/dr}, \tag{5}$$

where the sum is of values at the points where $V_g = - c\Delta f/f_0$. According to (1), the line profile $T_b(f)$ in this direction will then be the result of contributions from such points for every f. We should here note that near the galactic center and near strong sources it is necessary to subtract the contribution from continuous thermal radiation. Then, if a uniform temperature T_s as given by T_b in directions of greatest optical depth is adopted, the line profile in any direction yields $\tau(f)$, and hence the sum $\sum n/(dV_g/dr)$ can be determined for each value of f or V_g. In particular, if there is only one point of the trajectory where $V_g = - c\Delta f/f_0$ we can assign to it the value of $n/(dV_g/dr)$ corresponding to $\tau(f)$. Moreover, if the local value of dV_g/dr is known, the density of neutral hydrogen is determined at that point. It is on this principle that 21-cm models of the galactic distribution of neutral hydrogen have been made.

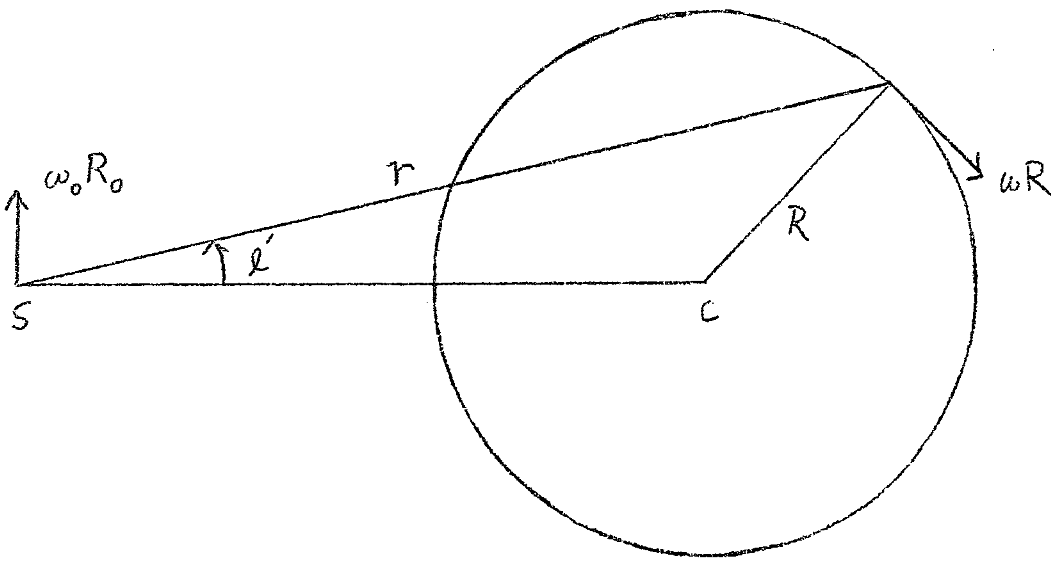

Figure 1. The motions in the galactic plane

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4.4 The distribution of neutral hydrogen in the galactic plane

In accordance with the generally accepted picture, we assume that the differential motion of the galactic material is one of differential rotation about the axis of galactic coordinates with angular velocity $\omega(R)$ in the negative sense, where R is the distance from the axis. If the existence of a "K-term" in the motion of the H I clouds is subsequently discovered, this analysis will have to be modified. In accordance with the evidence, we take $\omega(R)$ to be a monotonic decreasing function. Then if $\omega_0 = \omega(R_0)$ is the angular velocity of material in the neighborhood of the Sun $(R = R_0)$ it is a simple matter of kinematics to obtain Bottlinger's formula

$$V_g = R_0 \left\{ \omega(R) - \omega_0 \right\} \sin \ell' \qquad (1)$$

for the radial velocity of material distant r from the Sun in the galactic plane at longitude ℓ' relative to the center. In terms of distances from the center

$$R^2 = r^2 + R_0^2 - 2 r R_0 \cos \ell' . \qquad (2)$$

According to (1), V_g is also a monotonic decreasing function of R.

Neglecting any velocity dispersion we can say that the contributions to $F_b(f)$ from the direction ℓ' come from those points on the trajectory distant R from the center where V_g is equal to $- c\Delta f/f_0$. There are in general two such points, P_1, P_2, for each ℓ', at distances r_1, r_2 given by the solutions of (2).

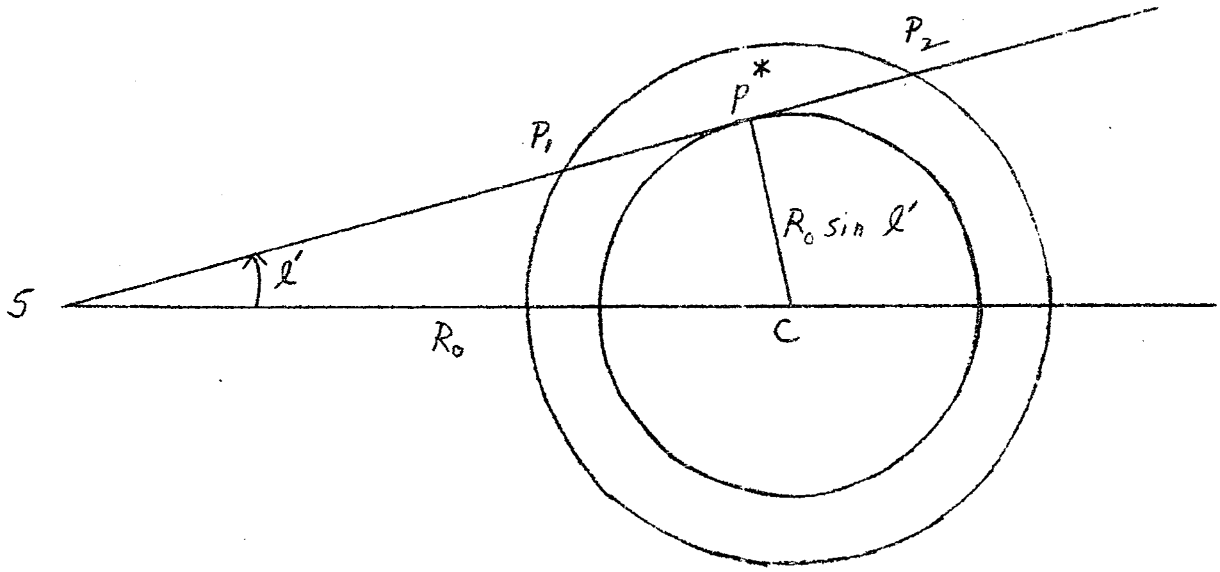

Figure 2. A ray trajectory in the range $0 < l' < 90°$

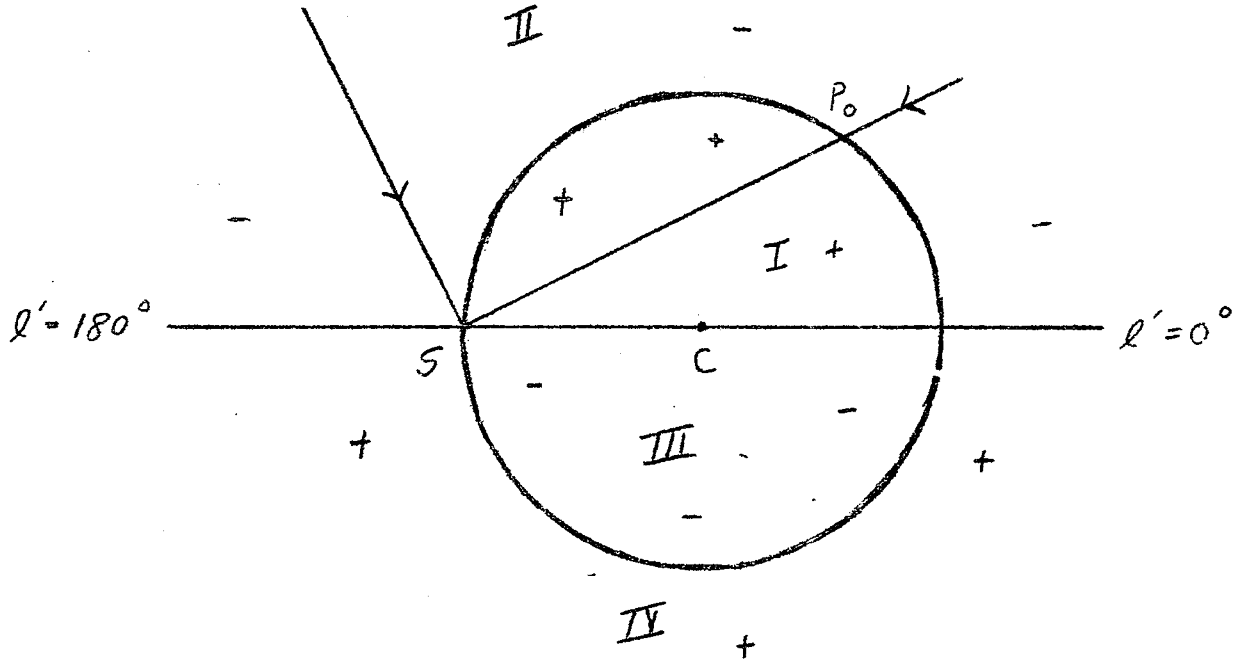

Figure 3. The zones of positive and negative V_g.

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$$\left.\begin{aligned} r_1 &= R_0 \cos \ell' + \sqrt{(R^2 - R_0{}^2 \sin^2 \ell')}, \\ r_2 &= R_0 \cos \ell' - \sqrt{(R^2 - R_0{}^2 \sin^2 \ell')}. \end{aligned}\right\} \qquad (3)$$

These become coincident at the point P^* where the trajectory is tangential to the circle $R = R^* = R_0 \sin \ell'$. Again there is only one such point in directions where $r_1 r_2 < 0$, i.e. where $R^2 > R_0{}^2$. Thus, in the range of longitudes $90° < |\ell'| < 180°$ there is a one-one correspondence between points P on the trajectory and the function $T_b(f)$ so that the value of $n/(dV_g/dr)$ can be assigned without ambiguity. If, further, $\omega(R)$ is known in the outer part of the Galaxy, (1) and (3) yield the formula

$$\frac{dV_g}{dr} = \pm R_0 \sin \ell' \sqrt{(R^2 - R_0{}^2 \sin^2 \ell')} \frac{d\omega}{dR} / R , \qquad (4)$$

whence n is determined.

According to (1), the radial velocity is zero along the lines $\ell' = 0°, 180°$ and on the circle $R = R_0$ through the Sun. These lines separate the zones I, IV where V_g is always positive from those II, III where V_g is negative. Since the radial velocities in zones III and IV are of the same magnitude but of opposite sign to those at the image points in zones I and II, it is sufficient to confine our discussion to rays coming from the directions $0° < \ell' < 180°$. For $90° < \ell' < 180°$ the trajectories lie entirely in zone II; $-V_g$ increases steadily from 0 to $R_0 \omega_0 \sin \ell'$ resulting in assignable contributions to the profile in the range

$$0 < \Delta f < f_o \frac{R_0 \omega_0}{c} \sin \ell'.$$

For $0 < \ell' < 90°$, assignable contributions in the same frequency range come from the points beyond P_0, which lie in zone II; but between S and P_0 each contribution $T_b(f)$ comes from the two points P_1, P_2 which lie in zone I and contribute in the range

$$-(f_0 - f^*) < \Delta f < 0,$$

where $f^* = f_0(1 - V_g^*/c)$ corresponds to the tangent point P^*. Now by (1)

$$V_g^* = \Theta(R^*) - \Theta_0 \sin \ell' , \qquad (5)$$

where $\Theta(R)$ is the rotational velocity at distance R and $\Theta_0 = \Theta(R_0)$. Provided neutral hydrogen is present at P the lowest frequency contributing to the profile may be taken as f^* and $T_b(f^*)$ assigned to P. Kwee, Muller and Westerhout (1954) have used the corresponding values of V_g^* in (5) to obtain the distribution of rotational velocity in the inner parts of the Galaxy. The corresponding distribution of dV_g/dr enables n to be determined at the various points P^* in zone II. In the two bounding directions $\ell' = 0°$ and 180°, towards the center and anticenter, there is no Doppler shift so that nothing can be inferred as to the distribution of neutral hydrogen. Here the extent to which the observed profiles depart from a sharp line will provide a measure of the velocity dispersions actually present, or of the deviation from the assumption of a differential motion of pure rotation. Likewise, the extensions of observed profiles from the directions $90° < |\ell'| < 180°$ to frequencies below f_0 can be similarly accounted for.

Schmidt (1957) has attempted to separate the contributions from points P_1 and P_2 in zone I by analyzing the distribution of T_b in latitude near the galactic plane. The limiting contributions $T_b(f^*)$ are assigned to points in a line through P^* perpendicular to the galactic plane. Since the distance r^* is equal to $R_0 \cos \ell'$ these can be converted into distributions with respect to $z = r^* b$, the distance above the plane. He found that the normalized z-distribution did not vary significantly with R, and hence assumed the same form $\phi(z - z_{max})$, normalized so that $\phi(0) = 1$, everywhere in the neighborhood of the galactic plane. Now at frequencies where both P_1 and P_2 contribute, the dependence of T_b on b may be written

$$T_b(b) = T_1 \phi\left\{r_1(b - b_1)\right\} + T_2 \phi\left\{r_2(b - b_2)\right\}, \qquad (6)$$

in which the unknowns T_1, T_2, b_1, b_2 may in principle be determined from the values of T_b at four different values of b. Thus $n_1/(dV_g/dr)_1$ and $n_2/(dV_g/dr)_2$ can be found and hence the densities n_1 and n_2 at P_1 and P_2. This can be done for each frequency in the range $-(f_0 - f^*) < \Delta f < 0$, so that the whose distribution of n along SP_0 can be determined.

4.5 Corrections

(a) Velocity dispersion. If the velocity dispersion is not negligible, as has so far been assumed, the equation Sec. 4.3 (5) for the optical depth of a trajectory for the frequency f requires modification. Suppose the distribution of cloud velocities is Gaussian, so that

$$N(V,\underset{\sim}{r}) = \frac{n(r)}{\eta\sqrt{(2\pi)}}\, e^{-(V - V_g)^2/2\eta^2}. \qquad (1)$$

Then, instead of Sec. 4.3 (5), Sec. 4.3 (3) gives

$$\tau(f) = \frac{\beta}{T_s\sqrt{(2\pi)}}\int_{-\infty}^{\infty} \frac{1}{\eta}\,\frac{n}{dV_g/dr}\,\exp\left\{-\left(c\frac{\Delta f}{f_0} + V_g\right)^2/2\eta^2\right\} dV_g, \qquad (2)$$

where by including η within the integrand we have allowed for the possibility of different dispersions for the different H I cloud groups contributing in the neighborhoods of the points P_1 and P_2. Where there is no ambiguity we may take η as constant; then (2) represents an integral equation for the determination of $n/(dV_g/dr)$ as a function of V_g, and hence of position on the trajectory. In the ambiguous case, the same can be done with each of the values τ_1, τ_2 once the contributions T_1, T_2 have been separated. The effect of the velocity dispersion is similar to that of aerial smoothing (Sec. 1.7), and again the restoration of the true values of $n/(dV_g/dr)$ from those given by $T_s\,\tau/\beta$ in Sec. 4.3(5) cannot be carried out completely. Corresponding to the "invisible" components of the brightness temperature, Fourier components of wave number greater than about $1/\eta$ are heavily attenuated by the Fourier transform of the distribution function, and in practice cannot be recovered.

<u>(b) Beamwidth.</u> It is, of course, necessary to take account of the effects of the finite beamwidth of the aerial, which for any frequency will have the effect particularly of broadening the distribution in galactic latitude about the galactic plane. For pencil-beam aerials of the order

of a few degrees between half-power points it is usually possible to fit a Gaussian radiation pattern, so that the restoration of T_b from the aerial-temperature distribution T_a involves the solution of a two-dimensional integral equation of a type similar to (2).

(c) Bandwidth. In the present case, where the profiles $T_b(f)$ provide the data from which the inferences are drawn, it is also necessary to correct for the broadening effect of the finite receiver bandwidth on the profiles. Again, it is usually possible to fit a Gaussian curve to the response over the bandwidth, so that after correction for beamwidth, the observed value $T_b'(f)$ is given in terms of the "true" value T_b by the equation

$$T_b'(f) = \frac{1}{\sigma\sqrt{(2\pi)}} \int_{-\infty}^{\infty} e^{-(x-f)^2/2\sigma^2}\, T_b(x)\, dx \,. \qquad (3)$$

Again this represents an integral equation for the determination of the true profile $T_b(f)$ given the observed profile $T_b'(f)$ and the dispersion modulus σ. Approximate methods of solution have been given by Bracewell (1955) and Ollengren and van de Hulst (1957).

(d) Solar motion and the Earth's orbital motion. In all observed profiles an overall correction for the Doppler shift due to the Sun's motion with a velocity close to 20 km/sec relative to the local group and the orbital motion of the Earth has to be made.

(e) Continuous background. For directions in the neighborhood of strong sources and the galactic center, allowance has to be made for contributions from continuous radiation at the same frequency. We consider the case of a single group of H I clouds in the line of sight. Let T_c be the brightness temperature of continuous radiation in this direction and let T_s and τ be the spin temperature and line optical thickness of the group. In general there will be continuum-emitting material on both sides of the group. Let T_c' be the continuum brightness temperature due to the sources beyond the group, and let τ_0 be the continuum optical thickness of the material between the group and the Sun. Then the continuum contributions to T_b from beyond the group, and between the group and the Sun, are $T_c' e^{-\tau}$ and $T_c - T_c'$, respectively, whereas the line contribution is $T_s(1 - e^{-\tau}) e^{-\tau_0}$. Summing, we have

$$T_b = T_c + (T_s e^{-\tau_0} - T_c')(1 - e^{-\tau}). \qquad (4)$$

The switching systems generally adopted are designed to measure the difference $T_b - T_c$. It can be seen that when the cloud spin temperature is lower than the brightness temperature of the continuum sources beyond, the line will manifest itself in absorption. Comparison of the absorption line profiles of a discrete source with those expected from the model distribution of H I clouds in this direction can yield information as to whether the source is extragalactic or not. The galactic H II regions may also be studied by this means.

In such studies it is essential to have reliable estimates of the spin temperature in the clouds. The processes which may determine T_s have recently been considered by Field (1958). He concludes that under conditions of low density and/or high radiation intensity collisions become less effective, so that T_s may differ significantly from the kinetic temperature T. The effect is particularly marked in the vicinity of strong sources.

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Chapter 5

Synchrotron Radiation

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Synchrotron Radiation

5.1. Introduction

The synchrotron process by which radiation is generated deserves careful study since it has virtually been established that it is the primary mechanism operating in the case of the discrete radio source Taurus A which was early identified with the Crab Nebula. This nebula is the expanding shell of a supernova explosion, probably that observed by the Chinese in A.D. 1054. Detailed observations of the optical radiation indicate that it is strongly polarized, evidence which supports Shklovsky's (1953) theory that the light from the Crab nebula is due to high-energy electrons spiralling in a magnetic field and emitting radiation by the synchrotron process. Oort and Walraven (1956) have estimated the field as of the order of 10^{-3} gauss. They found, too, that such electrons would produce sufficient radiation in the radio-frequency spectrum to account for the observed intensity. More recent studies by Pikelner (1956) and Woltjer (1958) indicate that 10^{-4} gauss is a better value for the field. Such a mechanism was first suggested for radio stars by Alfvén and Herlofson (1950). The synchrotron mechanism may well prove responsible for the high-intensity radiation received from many radio stars, which by reason both of their intensities and their spectra cannot be ascribed to a thermal origin. It may also prove to be the prime contributor to the "isotropic" component or "halo" in the background of galactic radiation.

5.2. The motion of a charged particle in a magnetic field

We consider the motion of a particle of rest mass m and charge e in a uniform magnetic field $\underset{\sim}{B}_0$. The equation of motion is

$$\frac{d}{dt}\frac{m\underset{\sim}{v}}{\sqrt{(1-\beta^2)}} = e\underset{\sim}{v}\wedge\underset{\sim}{B}_0 \ , \ \underset{\sim}{\beta}=\underset{\sim}{v}/c \ . \qquad (1)$$

It follows from this equation that β and hence the speed v is always constant. For, taking the scalar product of both sides with $m\underset{\sim}{v}/\sqrt{(1-\beta^2)}$, we get

$$\frac{d}{dt}\ \frac{m^2c^2\beta^2}{1-\beta^2} = 0 \ ,$$

whence β = constant. Thus (1) gives

$$\dot{\underset{\sim}{v}} = \frac{e}{m}\sqrt{(1-\beta^2)}\ \underset{\sim}{v}\wedge\underset{\sim}{B}_0 \ ,$$

or, in terms of the angular gyro-frequency vector

$$\underset{\sim}{\omega}_B = \frac{e}{m}\sqrt{(1-\beta^2)}\ \underset{\sim}{B}_0 \ , \qquad (2)$$

$$\dot{\underset{\sim}{v}} = -\underset{\sim}{\omega}_B\wedge\underset{\sim}{v} \ . \qquad (3)$$

It can be seen that the effect of having velocities comparable to c is to reduce the classical gyro-frequency $\omega_{B0} = |e|B_0/m$ by the factor $\sqrt{(1-\beta^2)}$.

The equation (3) may be considered in terms of components $\underset{\sim}{v}_{\parallel}$, $\underset{\sim}{v}_{\perp}$, parallel and perpendicular to $\underset{\sim}{B}_0$. We have

$$\dot{\underset{\sim}{v}}_{\parallel} = 0 \ ,$$

$$\dot{\underset{\sim}{v}}_{\perp} = -\underset{\sim}{\omega}_B\wedge\underset{\sim}{v} = -\underset{\sim}{\omega}_B\wedge\underset{\sim}{v}_{\perp} \ ,$$

from which we see that the motion consists of a uniform advance along the magnetic field with constant speed $v_{\parallel}$, together with a uniform

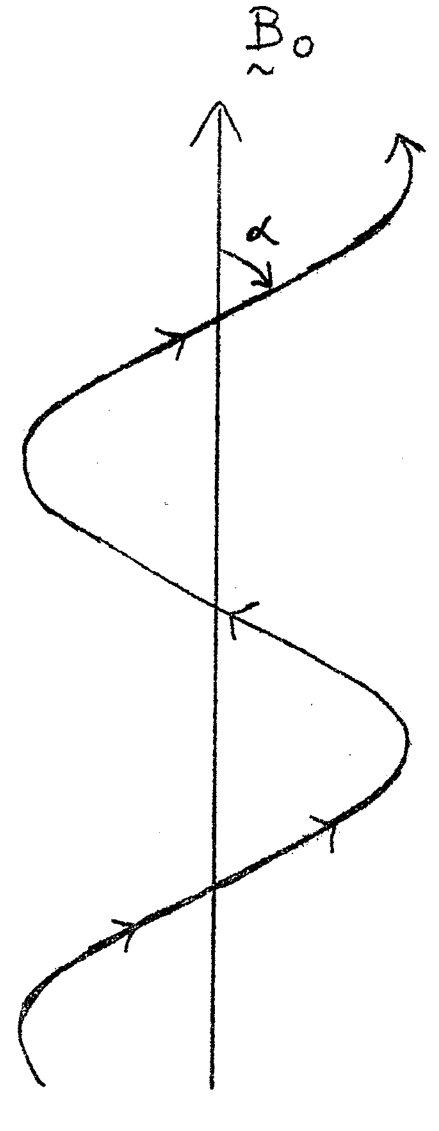

Figure 1. The electron trajectory

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circular motion of radius $v_\perp/\omega_B$ about the direction of $\underset{\sim}{B}_0$ with the angular velocity $-\underset{\sim}{\omega}_B$. We note that for an electron ($e<0$) the gyration will be in the RH sense about the direction of $\underset{\sim}{B}_0$, such as to reduce the intensity of $\underset{\sim}{B}_0$. If we take our origin on the axis of the helix at the projection from the initial position, the electron trajectory is given by the equation

$$\underset{\sim}{r} = \frac{v_\perp}{\omega_B}(\underset{\sim}{i}\cos\omega_B t + \underset{\sim}{j}\sin\omega_B t) + \underset{\sim}{k}v_\parallel t ,$$

where $\omega_B = |\underset{\sim}{\omega}_B|$, $\underset{\sim}{B}_0 = B_0\underset{\sim}{k}$ and $\underset{\sim}{i}$, $\underset{\sim}{j}$, are RH orthogonal unit vectors such that $\underset{\sim}{i}$ is in the direction of the initial position. Since $\underset{\sim}{v}$ makes a constant angle, α say, with $\underset{\sim}{k}$ we may write this

$$\underset{\sim}{r} = \frac{v}{\omega_B}\sin\alpha\,(\underset{\sim}{i}\cos\omega_B t + \underset{\sim}{j}\sin\omega_B t) + \underset{\sim}{k}v\cos\alpha\, t, \quad (4)$$

whence

$$\underset{\sim}{v} = v\underset{\sim}{\tau}, \quad (5)$$

where

$$\underset{\sim}{\tau} = \sin\alpha\,(-\underset{\sim}{i}\sin\omega_B t + \underset{\sim}{j}\cos\omega_B t) + \underset{\sim}{k}\cos\alpha . \quad (6)$$

We note that by (3) the acceleration is in the direction of $\underset{\sim}{k}\wedge\underset{\sim}{\tau}$ which is along the principal normal to the trajectory. The radius of curvature is equal to the constant $v/(\omega_B\sin\alpha)$.

5.3. The total radiation field

The total emission. Since in its motion the charge is continually accelerating, it is radiating in all directions at a rate $P(t)$, which for velocities $v \ll c$ is given by the classical formula

Landau + Lipschitz

$$P(t) = \frac{e^2\mu_v}{6\pi c} \quad \dot{v}^2 \; . \qquad (1)$$

To obtain a formula valid for velocities comparable to c we observe that P(t), being the energy radiated per unit time, represents the ratio of the timelike components of two four-vectors, the four-momentum

$$p_\alpha = \left(\frac{m\underset{\sim}{v}}{\sqrt{(1-\beta^2)}} , \frac{i\mathcal{E}}{c} \right) ,$$

where

$$\mathcal{E} = \frac{mc^2}{\sqrt{(1-\beta^2)}}$$

is the energy, and the position vector

$$x_\alpha = (\underset{\sim}{r}, \; ict) ,$$

and hence must be Lorentz invariant. Hence, if the above formula for P(t) can be put into Lorentz form, the result will be applicable in any frame of reference.

Now the 4-velocity and 4-acceleration vectors are obtained by differentiating x_α with respect to the proper time s, which is an invariant of spatial dimensions such that

$$ds^2 = c^2 dt^2 (1-\beta^2) \; .$$

Then the 4-velocity is given by

$$u_\alpha = \frac{dx_\alpha}{ds} = \frac{(\underset{\sim}{v}, \; ic)}{c\sqrt{(1-\beta^2)}} \; .$$

If $v \ll c$, the 4-acceleration du_α/ds is given approximately by

$$\frac{du_\alpha}{ds} = \frac{(\dot{\underset{\sim}{v}}, \; 0)}{c^2} \; .$$

The Lorentz-invariant form of (1) is therefore

$$P(t) = \frac{\mu_v e^2 c^3}{6\pi} \left(\frac{du_\alpha}{ds}\right)^2 . \qquad (2)$$

In the present case the 4-vector equation of motion is

$$mc \frac{du_\alpha}{ds} = eF_{\alpha\beta}\, u_\beta , \qquad (3)$$

where the components of the field tensor $F_{\alpha\beta}$ are given by

$$\left.\begin{aligned} F_{ij} &= \epsilon_{ijk}\, B_{0k} , \qquad i,j,k = 1,2,3, \\ F_{i4} &= F_{4i} = 0, \end{aligned}\right\} \qquad (4)$$

in which ϵ_{ijk} is the 3-dimensional alternating tensor. The spacelike part of (3) yields Sec. 5.2. (1) and the timelike part expresses the fact that the energy $\mathcal{E}$, and hence β, must remain constant. By (3) and (4)

$$\left(\frac{du_\alpha}{ds}\right)^2 = \left(\frac{e}{mc} \frac{\underset{\sim}{v} \wedge \underset{\sim}{B_0}}{c\sqrt{(1-\beta^2)}}\right)^2 ,$$

whence the total rate of radiation in all directions is given by $\omega_\beta = \frac{|e| B_0}{m}\sqrt{1-\beta^2}$

$$P(t) = \frac{\mu_v e^2 c}{6\pi} \frac{\omega_B{}^2 \beta^2 \sin^2 \alpha}{(1-\beta^2)^2} . \qquad (5)$$

For radiating electrons

$$P(t) = 1.58 \times 10^{-15}\, B_0{}^2 \sin^2 \alpha \frac{\beta^2}{1-\beta^2} \quad \text{ergs/sec,}$$

where B_0 is expressed in gauss.

The formula (5) indicates that $P(t)$ is a constant for a given speed, but increases according to the factor $\beta^2/(1-\beta^2)$ as v increases, being very great in the ultrarelativistic case where β is close to unity.

We therefore look to ultrarelativistic electrons gyrating in a magnetic field as an efficient source of non-thermal radiation. Since the electron executes a periodic motion it will radiate in harmonics of the fundamental frequency $f_B = \omega_B/2\pi$. For B_0 in gauss

$$f_B = 2.80\, B_0 \sqrt{(1-\beta^2)}\ \text{Mc/s} .$$

In the Crab nebula $B_0 \simeq 10^{-4}$ gauss and this probably diminishes to about 10^{-6} gauss in interstellar space. Clearly, if the synchrotron mechanism is responsible for the observed optical and radio emission, this must occur mainly in the very high harmonics of the fundamental frequency ω_B, where the spectrum is practically continuous.

Although, as can be readily verified, the rate of radiation is so small as not to affect the motion of the charge, over a long period of time an electron must have lost a significant proportion of its energy. Oort and Walraven calculate the half-life of an electron radiating by the synchrotron process. In terms of the energy $\mathcal{E}$ and the "classical radius" of the electron $a = \mu_v e^2/4\pi m$, we have

$$\frac{d\mathcal{E}}{dt} = -P(t) = -\frac{2a}{3c\,\mathcal{E}_0}\ \omega_{B0}^{\ 2}\ \sin^2\alpha\ (\mathcal{E}^2 - \mathcal{E}_0^{\ 2}), \quad (6)$$

where the subscript 0 denotes "rest" values of $\mathcal{E}$ and ω_B. On integration we get

$$\frac{1}{2}\ln\frac{\mathcal{E}+\mathcal{E}_0}{\mathcal{E}-\mathcal{E}_0} = \frac{2a}{3c}\,\omega_{B0}^{\ 2}\sin^2\alpha\ t + \text{const.}$$

We are interested in the case where $\mathcal{E} \gg \mathcal{E}_0$. Then, if $\mathcal{E}(0)$ denotes the initial value of the energy,

$$\frac{\mathcal{E}_0}{\mathcal{E}} - \frac{\mathcal{E}_0}{\mathcal{E}(0)} = \frac{2a}{3c}\,\omega_{B0}^2 \sin^2\alpha\, t\,.$$

It follows that the time $T_{\frac{1}{2}}$ for $\mathcal{E}$ to be reduced to $\frac{1}{2}\,\mathcal{E}(0)$ is given by

$$T_{\frac{1}{2}} = \frac{3c\,\mathcal{E}_0}{2a\,\mathcal{E}(0)\,\omega_{B0}^2 \sin^2\alpha}\,. \qquad (7)$$

The solution of (6) is now conveniently given as

$$\mathcal{E} = \frac{\mathcal{E}(0)}{1 + t/T_{\frac{1}{2}}}\,. \qquad (8)$$

Substitution of the values $a = 2.818 \times 10^{-13}$ cm , $\mathcal{E}_0 = 0.5110$ Mev into (7) gives, for $\mathcal{E}(0)$ expressed in Mev and B_0 in gauss,

$$T = 8.35/\mathcal{E}(0)B_0^2\sin^2\alpha \quad \text{years}\,.$$

Thus, if synchrotron radiation has been emitted by gyrating electrons in the Crab nebula since the year 1094, in a field $B_0 \simeq 10^{-4}$ gauss with relatively undiminished energies, the initial energies must have been of the order of 10^6 Mev, which is in the cosmic-ray range.

In order to determine the polarization of the radiation emitted, the direction of the electric vector must be found at any instant and in any direction. The Poynting flux vector then determines the angular distribution of the total emission. We shall again use the Parseval formulae of Fourier analysis to obtain the spectral distribution and polarization in terms of ω_B and β. Finally, we shall obtain formulae for the principal polarized components of the synchrotron emissivity of an electron gas in a magnetic field.

The Lienard-Wiechert Potentials. The total radiation field due to an accelerating charge is usually obtained via the Lienard-Wiechert potentials. The usual classical derivation of these, from the formulae for the retarded potentials of a distribution of charge and current, requires a delicate argument with respect to the different retarded times associated with the contributing elements as these are reduced to a point charge in motion. We show here that the derivation can be carried out in a straightforward manner by properly utilizing the properties of the "improper" delta function.

The retarded vector and scalar potentials $\underset{\sim}{A}, \varphi$ due to distributions of current and charge of density $\underset{\sim}{J}, \rho$ in free space are given by the formulae

$$\left.\begin{aligned} \underset{\sim}{A}(\underset{\sim}{r},t) &= \frac{\mu_v}{4\pi}\int \frac{\underset{\sim}{J}(\underset{\sim}{r}', t-R/c)}{R}\, dV', \\ \varphi(\underset{\sim}{r},t) &= \frac{1}{4\pi\epsilon_v}\int \frac{\rho(\underset{\sim}{r}', t-R/c)}{R}\, dV', \end{aligned}\right\} \qquad (9)$$

where R is the magnitude of the position of the field point $\underset{\sim}{r}$ relative to the source point $\underset{\sim}{r}'$,

$$\underset{\sim}{R} = \underset{\sim}{r} - \underset{\sim}{r}' , \qquad (10)$$

and $dV' = dV(\underset{\sim}{r}')$, i.e., the volume element surrounding the source point $\underset{\sim}{r}'$. Now if the source of the field is a charged particle whose position at time t is $\underset{\sim}{r}_1(t)$ and velocity $\underset{\sim}{v} = \dot{\underset{\sim}{r}}_1$, the source distributions may be represented as

$$\left.\begin{aligned} \underset{\sim}{J}(\underset{\sim}{r},t) &= e\underset{\sim}{v}(t)\ \delta(\underset{\sim}{r} - \underset{\sim}{r}_1(t)\ ), \\ \rho(\underset{\sim}{r},t) &= e\ \delta(\underset{\sim}{r} - \underset{\sim}{r}_1(t)\ )\quad . \end{aligned}\right\} \qquad (11)$$

In order to utilize the sifting property of the delta function we need to transform the variables of integration corresponding to the volume element $dV(\underset{\sim}{r}')$ to those corresponding to $dV(\underset{\sim}{r}^*)$, where $\underset{\sim}{r}^* = \underset{\sim}{r}' - \underset{\sim}{r}_1(t')$ and t' is the retarded time with respect to the charge,

$$t' = t - R(t')/c \quad . \tag{12}$$

Then

$$dV' = dV^* / \frac{\partial(\underset{\sim}{r}^*)}{\partial(\underset{\sim}{r}')} \; ,$$

where the latter term represents the Jacobian expressed in terms of the coordinates of $\underset{\sim}{r}^*$ and $\underset{\sim}{r}'$. It is easily seen that

$$\begin{aligned} \frac{\partial(\underset{\sim}{r}^*)}{\partial(\underset{\sim}{r}')} &= 1 - \underset{\sim}{v}'\cdot\underset{\sim}{R}(t')/cR(t') \\ &= 1 - \underset{\sim}{\beta}'\cdot\hat{\underset{\sim}{R}} \; , \end{aligned}$$

where $\hat{\underset{\sim}{R}}$ is the unit vector in the direction of $\underset{\sim}{R}(t')$. Thus we have

$$\underset{\sim}{A}(\underset{\sim}{r},t) = \frac{\mu_v e}{4\pi} \int \frac{\underset{\sim}{v}(t')\ \delta(\underset{\sim}{r}^*)}{R(t') - \underset{\sim}{\beta}'\cdot\underset{\sim}{R}(t')} \ dV^* \; ,$$

whence the Lienard-Wiechert vector potential

$$\underset{\sim}{A}(\underset{\sim}{r},t) = \frac{\mu_v e}{4\pi} \ \frac{\underset{\sim}{v}(t')}{R(t') - \underset{\sim}{\beta}'\cdot\underset{\sim}{R}(t')} \quad . \tag{13}$$

Similarly it can be shown that the scalar potential due to the moving charge is

$$\varphi(\underset{\sim}{r},t) = \frac{e}{4\pi\epsilon_v} \ \frac{1}{R(t') - \underset{\sim}{\beta}'\cdot\underset{\sim}{R}(t')} \quad . \tag{14}$$

<u>The Radiation Field Vectors</u> . The field vectors $\underset{\sim}{E}$, $\underset{\sim}{B}$ are obtained from the potentials by application of the formulae

$$\left.\begin{aligned} \underset{\sim}{E} &= -\frac{\partial A}{\partial t} - \text{grad}\ \varphi\ , \\ \underset{\sim}{B} &= \text{curl}\ \underset{\sim}{A}\ . \end{aligned}\right\} \tag{15}$$

For the purposes of calculation we introduce the quantity

$$s(t') = R(t') - \underset{\sim}{\beta}(t')\cdot\underset{\sim}{R}(t')\ . \tag{16}$$

Then we have

$$\frac{\partial s}{\partial t} = \frac{\partial s}{\partial t'}\ \frac{\partial t'}{\partial t} \tag{17}$$

and

$$\text{grad } s = \text{grad } R - \underset{\sim}{\beta}' + \frac{\partial s}{\partial t'}\ \text{grad } t'\ . \tag{18}$$

It is not difficult to show by differentiating the defining relation (12) that

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \underset{\sim}{\beta}'\cdot\underset{\sim}{\hat{R}}} = \frac{R}{s} \tag{19}$$

and

$$\text{grad } t' = -\frac{\hat{\underset{\approx}{R}}/c}{1-\underset{\sim}{\beta}'\cdot\underset{\sim}{\hat{R}}} = -\frac{\underset{\approx}{R}}{cs}\ . \tag{20}$$

Thus, in terms of s,

$$\underset{\sim}{A} = \frac{\mu_V ec}{4\pi}\frac{\underset{\sim}{\beta}'}{s}\ , \tag{21}$$

$$\frac{\varphi}{c} = \frac{\mu_V ec}{4\pi}\frac{1}{s}\ . \tag{22}$$

For the radiation field we retain only terms of order 1/s in the differentiations. Then, since

$$\frac{\partial s}{\partial t'} = -\dot{\beta}'\cdot R(1 + O(\tfrac{1}{s})),$$

$$\frac{\partial A}{\partial t} = \frac{\mu_v ec}{4\pi}\left(\frac{\dot{\beta}'}{s} + \frac{\beta'}{s^2}\dot{\beta}'\cdot R\right)\frac{R}{s},$$

$$\text{grad}\ \varphi = -\frac{\mu_v ec}{4\pi}\frac{\dot{\beta}'\cdot R}{s^2}\frac{R}{s},$$

$$\text{curl}\ A = -\frac{\mu_v ec}{4\pi}\frac{R}{cs}\wedge\left(\frac{\dot{\beta}'}{s} + \frac{\beta'}{s^2}\dot{\beta}'\cdot R\right),$$

and thus

$$E = \frac{\mu_v ec}{4\pi}\ \frac{(R - \beta' R)\ \dot{\beta}'\cdot R - \dot{\beta}' R s}{s^3}, \tag{23}$$

$$B = \frac{1}{c}\hat{R}\wedge E. \tag{24}$$

Substituting for s and collecting terms, we finally get for the electric vector of the radiation field

$$E = \frac{\mu_v ec}{4\pi R}\frac{\hat{R}\wedge\{(\hat{R} - \beta')\wedge\dot{\beta}'\}}{(1 - \beta'\cdot\hat{R})^3}. \tag{25}$$

<u>The Angular Distribution of the Total Radiation.</u> At the position r the radiant flux density at time t is given by the Poynting vector

$$S = E\wedge H, \tag{26}$$

where

$$H = B/\mu_v. \tag{27}$$

The total rate of radiation at the charge into the solid angle $d\Omega$.

about the direction $\hat{R}$ is $P(\hat{R},t')d\Omega$, such that

$$P(\hat{R},t')d\Omega dt' = S(r,t)\cdot\hat{R}\, dS\, dt ,$$

where

$$dS = R^2\, d\Omega \quad .$$

Since, by (24), (26), and (27),

$$S = \frac{1}{\mu_v c} E^2 \hat{R} \quad , \tag{28}$$

substitution from (25) gives

$$P(\hat{R},t') = \frac{\mu_v e^2 c}{16\pi^2} \frac{\{(\hat{R} - \beta')\dot{\beta}'\cdot\hat{R} - \dot{\beta}'(1 - \beta'\cdot\hat{R})\}^2}{(1 - \beta'\cdot\hat{R})^6} \frac{dt}{dt'} \quad .$$

By (19) this gives for the power radiated per unit solid angle about the direction of n from a charge moving with velocity βc

$$P(n,t) = \frac{\mu_v e^2 c}{16\pi^2} \frac{\{(n - \beta)\dot{\beta}\cdot n - \dot{\beta}(1 - \beta\cdot n)\}^2}{(1 - \beta\cdot n)^5} \quad . \tag{29}$$

For ultrarelativistically moving charges $1 - \beta^2 \ll 1$, so that the bulk of the radiation is emitted in the the direction of motion τ. Writing

$$\xi = \sqrt{(1 - \beta^2)} \tag{30}$$

and ϑ for the angle between n and τ, we have

$$\beta = 1 - \tfrac{1}{2}\xi^2 + O(\xi^4) \ ,$$

$$1 - \beta\cdot n = 1 - \cos\vartheta + \tfrac{1}{2}\xi^2 \cos\vartheta + O(\xi^4) \ ,$$

so that

$$P(\underset{\sim}{n},t) = \frac{\mu_v e^2 c}{16\pi^2} \frac{\{(\underset{\sim}{n}-\underset{\sim}{\tau})\dot{\underset{\sim}{\beta}}\cdot\underset{\sim}{n} - \dot{\underset{\sim}{\beta}}(1-\cos\vartheta) + \frac{1}{2}\xi^2(\underset{\sim}{\tau}\dot{\underset{\sim}{\beta}}\cdot\underset{\sim}{n} - \dot{\underset{\sim}{\beta}}\cos\vartheta)\}^2}{(1-\cos\vartheta + \frac{1}{2}\xi^2\cos\vartheta)^5},$$

approximately. Thus, for directions such that $\vartheta \gg \xi$, the ratio

$$\frac{P(\underset{\sim}{n},t)}{P(\underset{\sim}{\tau},t)} = O(\xi^6) \quad ,$$

which is negligibly small. For ϑ small, $1 - \underset{\sim}{\beta}\cdot\underset{\sim}{n} = \frac{1}{2}(\xi^2 + \vartheta^2)$, so that the denominator remains small as long as $\vartheta = O(\xi)$, which may be taken as the angular dimension of the cone of emission about the instantaneous direction of motion. This conclusion is confirmed by a more precise calculation. Since $|\underset{\sim}{n} - \underset{\sim}{\tau}| = O(\vartheta)$ the expression in the braces is equal to

$$(\underset{\sim}{n}-\underset{\sim}{\tau})\dot{\underset{\sim}{\beta}}\cdot\underset{\sim}{\tau} + (\underset{\sim}{n}-\underset{\sim}{\tau})\dot{\underset{\sim}{\beta}}\cdot(\underset{\sim}{n}-\underset{\sim}{\tau}) - \frac{1}{2}\vartheta^2\dot{\underset{\sim}{\beta}} - \frac{1}{2}\xi^2(\dot{\underset{\sim}{\beta}} - \dot{\underset{\sim}{\beta}}\cdot\underset{\sim}{\tau}\,\underset{\sim}{\tau})$$

plus third-order terms in ξ and ϑ. Now in the present case of a gyrating electron we saw in Sec. 5.2 that

$$\dot{\underset{\sim}{\beta}} = \omega_B \beta\, \underset{\sim}{k} \wedge \underset{\sim}{\tau}. \qquad (31)$$

Hence $\dot{\underset{\sim}{\beta}}\cdot\underset{\sim}{\tau} = 0$ and, after some algebra, the expression for $P(\underset{\sim}{n},t)$ reduces to

$$P(\underset{\sim}{n},t) = \frac{\mu_v e^2 c}{2\pi^2}\, \omega_B^2 \sin^2\alpha\, \frac{(\xi^2 + \vartheta^2)^2 - 4\xi^2\vartheta^2\sin^2\varphi}{(\xi^2 + \vartheta^2)^5}, \qquad (32)$$

where φ is the azimuthal angle of $\underset{\sim}{n}$ relative to $\underset{\sim}{\tau}$, measured from the meridian through the direction of $\underset{\sim}{k}$.

To the same order of approximation as in (32), (5) gives

$$P(t) = \frac{\mu_v e^2 c}{6\pi} \frac{\omega_B^2 \sin^2\alpha}{\xi^4} \qquad (33)$$

for the total power radiated in all directions. Now the total power radiated within an angle ϑ of the direction of motion ξ is

$$P(\vartheta, t) = \int_0^{\vartheta}\int_0^{2\pi} P(\underset{\sim}{n}, t)\, d\varphi\, \vartheta\, d\vartheta .$$

Hence the fraction radiated within an angle ϑ is

$$\frac{P(\vartheta, t)}{P(t)} = 6\int_0^{\vartheta} \frac{(1 + \vartheta^2/\xi^2)^2 - 2\vartheta^2/\xi^2}{(1 + \vartheta^2/\xi^2)^5} \frac{\vartheta\, d\vartheta}{\xi^2} .$$

On performing the integration we get

$$\frac{P(\vartheta, t)}{P(t)} = 1 - \frac{3u^2 - 4u + 3}{2u^4} , \quad u = 1 + \vartheta^2/\xi^2 , \qquad (34)$$

from which we construct the following table.

TABLE 1

$1 - P(\vartheta, t)/P(t)$

ϑ/ξ	0	1	2	3	5	10
u	1	2	5	10	26	101
$1 - P(\vartheta, t)/P(t)$	1	0.22	0.046	0.012	0.0052	0.0003

From these values it is clear that effectively the radiation is all emitted within an angular distance of a few multiples of $\xi = \sqrt{(1 - \beta^2)}$ from the direction of motion.

$W = \frac{m_0 c^2}{\xi}$

$\xi = \left(\frac{m_0 c^2}{W}\right) \approx \left(\frac{m_0 c^2}{E}\right)$

5.4. The Spectrum of the Radiation Field

The fact that radiation is received from an accelerating charge moving with an ultrarelativistic velocity, only during the small time it takes for the cone of emission to sweep across the direction to the field point, enables us to estimate the spectral range of the radiation emitted. At the charge, the time taken for the direction of motion $\underset{\sim}{\tau}(t')$ to sweep through the small angle ξ is

$$\Delta t' = \xi / |\dot{\underset{\sim}{\tau}}| = \xi / \kappa v,$$

where κ is the instantaneous value of the curvature. The corresponding time interval Δt at the field point is diminished by the factor $\partial t/\partial t'$, so that by Sec. 5.3 (19)

$$\Delta t = (1 - \underset{\sim}{\beta}'\cdot\hat{\underset{\sim}{R}})\,\xi/\kappa v,$$
$$= \frac{1}{2}(\xi^2 + \vartheta^2)\xi/\kappa c$$

on introducing the approximations of the previous section. Hence the radiation is received in pulses of effective duration $\Delta t \sim \xi^3/\kappa c$. It follows that the spectrum of the radiation received contains frequencies up to the large value $\omega_c \simeq \kappa c/\xi^3$.

For the case of a gyrating electron Sec. 5.2 (6) gives

$$|\dot{\underset{\sim}{\tau}}| = \omega_B \sin\alpha ,$$

where

$$\omega_c \simeq \omega_B \sin\alpha / \xi^3.$$

The distribution of energy over the spectrum depends, of course, on the circumstances of the motion of the charge.

<u>The field spectrum</u>. The amplitude $\underset{\sim}{E}_n(\underset{\sim}{r})$ of the n-th harmonic of the field

$$\underset{\sim}{E}(\underset{\sim}{r},t) = \sum_{-\infty}^{\infty} \underset{\sim}{E}_n\, e^{-in\omega_B t} \qquad (1)$$

due to a charge executing a periodic motion of fundamental period $2\pi/\omega_B$

is given by

$$\underset{\sim}{E}_n(\underset{\sim}{r}) = \frac{\omega_B}{2\pi}\int_0^{2\pi/\omega_B} \underset{\sim}{E} e^{in\omega_B t}dt , \qquad (2)$$

where $\underset{\sim}{E}$ is given by Sec. 5.3 (25). The evaluation of this integral is facilitated by the two circumstances that the distance $R(t')$ from the charge is large compared with the dimension of the region in which the motion takes place, and that the motion is ultrarelativistic.

In the first place we may take the reference origin close to the position of the charge during the time of observation. Then we have

$$\underset{\sim}{R}(t') = \underset{\sim}{r} - \underset{\sim}{r}_1(t') , \quad r_1 << r,$$

so that

$$R = r(1 - \underset{\sim}{r}_1 \cdot \underset{\sim}{r}/r^2) ,$$

or, writing $\underset{\sim}{n}$ for the unit vector in the direction of $\underset{\sim}{r}$,

$$\left.\begin{aligned} R &= r - \underset{\sim}{n}\cdot\underset{\sim}{r}_1 , \\ \hat{\underset{\sim}{R}} &= \underset{\sim}{n} . \end{aligned}\right\} \qquad (3)$$

With these approximations, and transforming to the particle time $t' = t - R(t')/c$ in (2), we get

$$\underset{\sim}{E}_n = \frac{\omega_B}{2\pi} e^{in\omega_B r/c} \int_0^{2\pi/\omega_B} \underset{\sim}{E}\, e^{in\omega_B(t' - \underset{\sim}{n}\cdot\underset{\sim}{r}_1/c)} \frac{\partial t}{\partial t'} dt' ,$$

$$= \frac{\mu_v ec}{8\pi^2 r}\omega_B e^{in\omega_B r/c} \int_0^{2\pi/\omega_B} \frac{\underset{\sim}{n}\wedge\{(\underset{\sim}{n} - \underset{\sim}{\beta}')\wedge\dot{\underset{\sim}{\beta}}'\}}{(1 - \underset{\sim}{\beta}'\cdot\underset{\sim}{n})^2} e^{in\omega_B(t' - \underset{\sim}{n}\cdot\underset{\sim}{r}_1/c)}dt' \qquad (4)$$

on substitution from Sec. 5.3 (19) and (25). Again, for ultrarelativistically gyrating electrons we saw in the previous section that the numerator

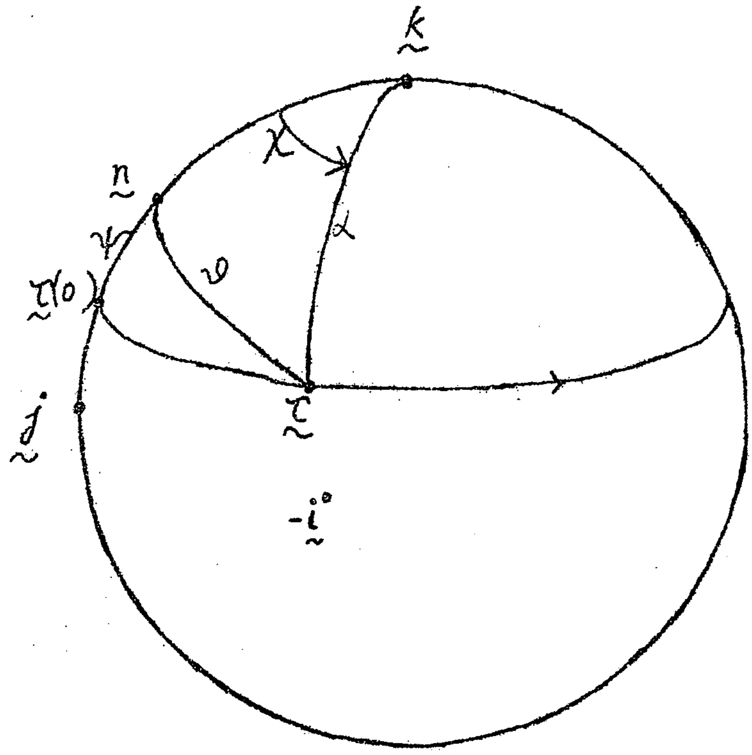

Figure 2. The directions of motion and of observation relative to the magnetic field

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was a small quantity of the second order in ξ and ϑ

$$\underset{\sim}{n} \wedge \left\{(\underset{\sim}{n} - \underset{\sim}{\beta}') \wedge \dot{\underset{\sim}{\beta}}'\right\} = (\underset{\sim}{n} - \underset{\sim}{\tau})\dot{\beta}' \cdot (\underset{\sim}{n} - \underset{\sim}{\tau}) - \frac{1}{2}(\xi^2 + \vartheta^2)\dot{\underset{\sim}{\beta}}',$$

where

$$\dot{\underset{\sim}{\beta}} = \omega_B \underset{\sim}{k} \wedge \underset{\sim}{\tau}.$$

Similarly, the denominator is of the fourth order

$$(1 - \underset{\sim}{\beta}' \cdot \underset{\sim}{n})^2 = \frac{1}{4}(\xi^2 + \vartheta^2)^2 .$$

Since for the purposes of observation we are interested only in the average radiant power received at any point, we may take $\underset{\sim}{n}$ in the plane containing the directions of the field $\underset{\sim}{B}_0$ and the initial velocity of the charge. Then if ψ is the angle between $\underset{\sim}{n}$ and $\underset{\sim}{\tau}(0)$, in the sense towards $\underset{\sim}{k}$ as in Figure 2, and $\chi = \omega_B t'$,

$$\underset{\sim}{n} = \underset{\sim}{j} \sin(\alpha - \psi) + \underset{\sim}{k} \cos(\alpha - \psi),$$

and

$$\underset{\sim}{r}_1 = \frac{v}{\omega_B}\left\{\sin\alpha\,(\underset{\sim}{i}\cos\chi + \underset{\sim}{j}\sin\chi) + \underset{\sim}{k}\chi\cos\alpha\right\},$$

$$\underset{\sim}{\tau} = \sin\alpha\,(-\underset{\sim}{i}\sin\chi + \underset{\sim}{j}\cos\chi) + \underset{\sim}{k}\cos\alpha ,$$

by Sec. 5.2 (4) and (6). We have seen that contributions to the integral from points corresponding to values of ϑ greater than $O(\xi)$ are insignificant. Hence we may replace the integrand by its approximate value for small ψ and χ and replace the terminals by $\pm\infty$. Then, to first order, we get

$$\underset{\sim}{n} - \underset{\sim}{\tau} = \underset{\sim}{i}\chi\sin\alpha - \psi(\underset{\sim}{j}\cos\alpha - \underset{\sim}{k}\sin\alpha),$$

$$\underset{\sim}{k} \wedge \underset{\sim}{\tau} = -\sin\alpha\,(\underset{\sim}{i} + \underset{\sim}{j}\chi),$$

and

$$\vartheta^2 = \psi^2 + \chi^2 \sin^2\alpha \ .$$

The exponent turns out to be a small quantity of the third order

$$\omega_B(t' - \underset{\sim}{n}\cdot\underset{\sim}{r}_1/c) = \frac{1}{2}(\xi^2 + \psi^2)\chi + \frac{1}{6}\chi^3 \sin^2\alpha \ .$$

Substituting from these approximations into (4) we get

$$\underset{\sim}{E}_n = \frac{\mu_v ec\,\omega_B \sin\alpha}{4\pi^2 r} e^{in\omega_B r/c} \int_{-\infty}^{\infty} e^{\frac{1}{2}in\chi(\xi^2 + \psi^2 + \frac{1}{3}\chi^2 \sin^2\alpha)}$$

$$\times \frac{(\xi^2 + \psi^2 - \chi^2 \sin^2\alpha)\underset{\sim}{i} + 2\psi\chi\sin\alpha(\underset{\sim}{j}\cos\alpha - \underset{\sim}{k}\sin\alpha)}{(\xi^2 + \psi^2 + \chi^2 \sin^2\alpha)^2} \, d\chi \ , \quad (5)$$

whose form suggests the possibility of expressing $\underset{\sim}{E}_n$ in terms of Airy functions Ai (x) or modified Bessel functions $K_\nu(x)$ of order 1/3. The Airy function is defined by the integral

$$\mathrm{Ai}(x) = \frac{1}{\pi}\int_0^{\infty} \cos\left(xu + \frac{1}{3}u^3\right) du$$

$$= \frac{1}{\pi\sqrt{3}} x^{\frac{1}{2}} K_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right),$$

and the integrals involved in (5) are of the form

$$\phi_1(\eta) = \int_{-\infty}^{\infty} e^{i\gamma(\eta^2 u + \frac{1}{3}u^3)} \frac{2u}{(\eta^2 + u^2)^2} \, du \ ,$$

$$\phi_2(\eta) = \int_{-\infty}^{\infty} e^{i\gamma(\eta^2 u + \frac{1}{3}u^3)} \frac{\eta^2 - u^2}{(\eta^2 + u^2)^2} \, du \ .$$

The first integral is easily resolved by observing that the second factor in the integrand is proportional to the derivative of the reciprocal of the derivative of the exponent. Then, on integration by parts, we get

$$\phi_1(\eta) = i\gamma F_\gamma(\eta) \quad ,$$

where

$$F_\gamma(\eta) = \int_{-\infty}^{\infty} e^{i\gamma(\eta^2 u + \frac{1}{3}u^3)} \, du \quad .$$

Again, on applying the same procedure to the integral whose integrand is u times that of $F_1(\eta)$ we get

$$\phi_2(\eta) = -\frac{1}{2\eta} F_\gamma{}'(\eta) \quad .$$

Thus (5) becomes

$$\underset{\sim}{E}_n = \frac{\mu_v ec}{4\pi^2 r} e^{in\omega_0 r/c} \left\{ -\frac{1}{2\eta} F_\gamma{}'(\eta)\underset{\sim}{i} + i\gamma\psi F_\gamma(\eta)(\underset{\sim}{j} \cos\alpha - \underset{\sim}{k} \sin\alpha) \right\}, \quad (6)$$

where

$$\eta = \sqrt{(\xi^2 + \psi^2)}, \; \gamma = n/(2 \sin\alpha) \quad .$$

Now

$$F_\gamma(\eta) = 2\pi\gamma^{-\frac{1}{3}} Ai(\gamma^{\frac{2}{3}}\eta^2)$$

$$= \frac{2}{\sqrt{3}} \eta \, K_{\frac{1}{3}}\left(\frac{2}{3}\gamma\eta^3\right).$$

It is more satisfactory to use the representation in terms of Bessel functions, which offers the advantages of various recurrence relations. In particular,

$$F_\gamma'(\eta) = \frac{2}{\sqrt{3}}\left(K_{\frac{1}{3}} + 2\gamma\eta^3 K_{\frac{1}{3}}' \right)$$

$$= -\frac{4}{\sqrt{3}}\gamma\eta^3 K_{\frac{2}{3}}\left(\frac{2}{3}\gamma\eta^3\right)$$

since $K_{\frac{1}{3}}'(x) + \frac{1}{3x} K_{\frac{1}{3}}(x) = -K_{\frac{2}{3}}(x)$.

Hence (6) becomes

$$\underset{\sim}{E}_n = \frac{\mu_v e c \omega_B}{4\sqrt{3}\pi^2 r} e^{in\omega_B r/c} \frac{n}{\sin\alpha} \left\{ (\xi^2 + \psi^2) K_{\frac{2}{3}}\left(\frac{n}{3\sin\alpha}(\xi^2+\psi^2)^{\frac{3}{2}}\right)\underset{\sim}{i} \right.$$

$$\left. + i\psi(\xi^2+\psi^2)^{\frac{1}{2}} K_{\frac{1}{3}}\left(\frac{n}{3\sin\alpha}(\xi^2+\psi^2)^{\frac{3}{2}}\right)(\underset{\sim}{j}\cos\alpha - \underset{\sim}{k}\sin\alpha)\right\} , \quad (7)$$

which is a function of the angle ψ between the direction of observation $\underset{\sim}{n}$ and the closest generator of the cone containing the directions of motion $\underset{\sim}{\zeta}$. Since, to a first approximation

$$\underset{\sim}{j}\cos\alpha - \underset{\sim}{k}\sin\alpha = \underset{\sim}{n}\wedge\underset{\sim}{i} ,$$

the ratio of the two terms within the braces determines the complex polarization of the n-th harmonic

$$Q_n = -(\underset{\sim}{E}_n\cdot\underset{\sim}{i})/(\underset{\sim}{E}_n\cdot\underset{\sim}{n}\wedge\underset{\sim}{i}) ,$$

which determines the characteristics of the polarization ellipse (cf. Sec. 2.3). Thus

$$Q_n(\psi) = i \frac{(\xi^2+\psi^2)^{\frac{1}{2}}}{\psi} \frac{K_{2/3}\left(\frac{n}{3\sin\alpha}(\xi^2+\psi^2)^{3/2}\right)}{K_{1/3}\left(\frac{n}{3\sin\alpha}(\xi^2+\psi^2)^{3/2}\right)} , \quad (8)$$

from which, since both Bessel functions are real positive, we conclude (cf. Westfold, 1958) that the polarization is elliptic, with the axes of the ellipse along directions parallel and perpendicular to the projection of $\underset{\sim}{B}_0$ on the plane transverse to $\underset{\sim}{n}$. The ratio of the axes is given by $|Q_n|$, with the major axis along the direction of $\underset{\sim}{n} \wedge \underset{\sim}{i}$ or $\underset{\sim}{i}$ according as $|Q_n| \lessgtr 1$: the sense of description of the ellipse is RH or LH according as $\psi \gtrless 0$. Oort and Walraven's (1956) statement that the polarization is linear with the electric vector parallel to the direction of $k \wedge \underset{\sim}{\zeta}(0)$, i.e. parallel to $\underset{\sim}{i}$, is seen to be true only when $\psi = 0$.

Since these results depend only on the motion while $\underset{\sim}{\zeta}$ is within a small neighborhood of $\underset{\sim}{n}$, they are also applicable to the radiation from any charge whose acceleration is instantaneously perpendicular to its velocity. The periodicity of the motion in the present case can have no significance. In confirmation of this conclusion, we shall see that most of the radiation is emitted in the spectral band near $\omega = \omega_c$. In this region $n = O(\xi^{-3})$ so that the line spectrum has become practically continuous.

<u>The spectral distribution of the emission.</u> Since the emitting charge executes periodic motion, the radiated power is distributed among the harmonics of the fundamental frequency. Being interested only in the average power radiated over a period, we may apply Parseval's theorem to the formula Sec. 5.3 (28) for the Poynting vector. Then the average

$$\bar{\underset{\sim}{S}} = \frac{\omega_B}{2\pi}\int_0^{2\pi/\omega_B} \underset{\sim}{S}\, dt$$

$$= \frac{\omega_B}{2\pi}\frac{\hat{\underset{\sim}{R}}}{\mu_v c}\int_0^{2\pi/\omega_B} E^2\, dt$$

$$= \frac{\hat{\underset{\sim}{R}}}{\mu_v c}\left(E_0^{\,2} + 2\sum_1^{\infty}|E_n|^2\right).$$

If $\overline{P}(\underset{\sim}{n})d\Omega(\underset{\sim}{n})$ is the average power radiated into the solid angle $d\Omega(\underset{\sim}{n})$ and we write

$$\overline{P}(\underset{\sim}{n}) = \frac{1}{2}\overline{P_0}(\underset{\sim}{n}) + \sum_{1}^{\infty} P_n(\underset{\sim}{n}) ,$$

then, relating $\overline{P}$ to $\underset{\sim}{\overline{S}}\cdot\underset{\sim}{\hat{R}}$ as before, the average power in the n-th harmonic radiated into the solid angle $d\Omega(\underset{\sim}{n})$ is $\overline{P_n}(\underset{\sim}{n})d\Omega(\underset{\sim}{n})$, where

$$\overline{P_n}(\underset{\sim}{n}) = \frac{2}{\mu_v c}\left|E_n\right|^2 r^2 . \tag{9}$$

It can be seen from (7) that $\overline{P_n}(\underset{\sim}{n})$ can be resolved into two contributions corresponding to the components of $\underset{\sim}{E}_n$ parallel to the directions of $\underset{\sim}{i}$ and $\underset{\sim}{n}\wedge\underset{\sim}{i}$, for all values of ψ. We shall distinguish such component contributions by the superscripts (2), (1) respectively, representing intensity components perpendicular and parallel to the projection of $\underset{\sim}{B}_0$ in the plane normal to $\underset{\sim}{n}$. Thus

$$\left.\begin{aligned} \overline{P_n^{(1)}}(\underset{\sim}{n}) &= \frac{\mu_v e^2 c\,\omega_B^2}{24\pi^4}\frac{n^2}{\sin^2\alpha}\psi^2(\xi^2+\psi^2)\,K_{\frac{1}{3}}^2\left(\frac{n}{3\sin\alpha}(\xi^2+\psi^2)^{3/2}\right), \\ \overline{P_n^{(2)}}(\underset{\sim}{n}) &= \frac{\mu_v e^2 c\,\omega_B^2}{24\pi^4}\frac{n^2}{\sin^2\alpha}(\xi^2+\psi^2)^2\,K_{\frac{2}{3}}^2\left(\frac{n}{3\sin\alpha}(\xi^2+\psi^2)^{3/2}\right) . \end{aligned}\right\} \tag{10}$$

The power radiated in the n-th harmonic is effectively confined to a small range $O(\xi)$ of ψ. Hence, in calculating the average power radiated in all directions in the n-th harmonic we may take

$$\begin{aligned} \overline{P_n} &= \int \overline{P_n}(\underset{\sim}{n})\, d\Omega \\ &= 2\pi\sin\alpha\int_{-\infty}^{\infty}\overline{P_n}(\underset{\sim}{n})\, d\psi . \end{aligned} \tag{11}$$

In calculating $\overline{P_n^{(1)}}$ and $\overline{P_n^{(2)}}$ it is simplest to start from the more primitive form (6). Then

$$\overline{P_n^{(1)}}(\underset{\sim}{n}) = \frac{\mu_v e^2 c\,\omega_B^2}{8\pi^4}\gamma^2\psi^2 \iint_{-\infty}^{\infty} e^{i\gamma\{\eta^2(u-v)+(u^3-v^3)\}}\,du\,dv,$$

$$\overline{P_n^{(2)}}(\underset{\sim}{n}) = \frac{\mu_v e^2 c\,\omega_B^2}{8\pi^4}\gamma^2 \iint_{-\infty}^{\infty} e^{i\gamma\{\eta^2(u-v)+(u^3-v^3)\}}uv\,du\,dv\ .$$

It is convenient to change the variables of integration u, v to x, y, given by

$$2x = u - v,\ 2y = u + v\ .$$

Then

$$\overline{P_n^{(1)}}(\underset{\sim}{n}) = \frac{\mu_v e^2 c\,\omega_B^2}{4\pi^4}\gamma^2\psi^2 \int_{-\infty}^{\infty} e^{2i\gamma(\eta^2 x+\frac{1}{3}x^3)}dx \int_{-\infty}^{\infty} e^{2i\gamma xy^2}\,dy\ ,$$

$$\overline{P_n^{(2)}}(\underset{\sim}{n}) = \frac{\mu_v e^2 c\,\omega_B^2}{4\pi^4}\gamma^2 \int_{-\infty}^{\infty} e^{2i\gamma(\eta^2 x+\frac{1}{3}x^3)}dx \int_{-\infty}^{\infty} e^{2i\gamma xy^2}(y^2-x^2)\,dy\ .$$

Since

$$\int_{-\infty}^{\infty} e^{2i\gamma xy^2}\,dy = e^{\pi i/4}\sqrt{\pi}(2\gamma x)^{-\frac{1}{2}}\ ,$$

$$\int_{-\infty}^{\infty} e^{2i\gamma xy^2}\,y^2\,dy = -\frac{1}{2}e^{-\pi i/4}\sqrt{\pi}(2\gamma x)^{-\frac{3}{2}}\ ,$$

in which the principal value is denoted by the fractional powers of x, we get the integral formulae

$$\overline{P_n^{(1)}}(\eta) = \frac{\mu_v e^2 c\,\omega_B^2}{16\pi^3\sqrt{\pi}}\,(2\gamma)^{3/2}\,e^{\pi i/4}\psi^2\int_{-\infty}^{\infty} e^{2i\gamma(\eta^2 x+\frac{1}{3}x^3)}\,x^{-\frac{1}{2}}\,dx\ ,$$

$$\overline{P_n^{(2)}}(\eta) = -\frac{\mu_v e^2 c\,\omega_B^2}{16\pi^3\sqrt{\pi}}\,(2\gamma)^{3/2}\,e^{\pi i/4}\int_{-\infty}^{\infty} e^{2i\gamma(\eta^2 x+\frac{1}{3}x^3)}\,(x^{1/2}+x^{-3/2}/4i\gamma)\,dx\ ,$$

which are to be interpreted as Cauchy principal values in respect of the singularities of the integrands at $x = 0$. We now substitute from these formulae into (11), recalling that $\eta^2 = \xi^2 + \psi^2$. Then the integrations with respect to ψ yield the results

$$\overline{P_n^{(1)}} = -i\,\frac{\mu_v e^2 c\,\omega_B^2\sin\alpha}{8\pi^2}\,\gamma\int_{-\infty}^{\infty} e^{2i\gamma(\xi^2 x+\frac{1}{3}x^3)}\,x^{-2}\,dx/2i\gamma\ ,$$

$$\overline{P_n^{(2)}} = -i\,\frac{\mu_v e^2 c\,\omega_B^2\sin\alpha}{8\pi^2}\,2\gamma\int_{-\infty}^{\infty} e^{2i\gamma(\xi^2 x+\frac{1}{3}x^3)}\,x\,dx + \overline{P_n^{(1)}}\,.$$

These integrals can be expressed in terms of Bessel functions in the following manner. Integration by parts gives

$$\int_{-\infty}^{\infty} e^{2i\gamma(\xi^2 x+\frac{1}{3}x^3)}\,x^{-2}\,dx/2i\gamma = \int_{-\infty}^{\infty} e^{2i\gamma(\xi^2 x+\frac{1}{3}x^3)}\,\frac{\xi^2+x^2}{x}\,dx\ .$$

Now

$$\int_{-\infty}^{\infty} e^{2i\gamma(\xi^2 x+\frac{1}{3}x^3)}\,x\,dx = \frac{1}{4i\gamma\xi}\,F_{2\gamma}'(\xi) = \frac{2i}{\sqrt{3}}\,\xi^2\,K_{\frac{2}{3}}(\tfrac{4}{3}\gamma\xi^3)$$

and

$$\frac{d}{d\xi}\int_{-\infty}^{\infty} e^{2i\gamma(\xi^2 x+\frac{1}{3}x^3)}\,x^{-1}\,dx = 4i\gamma\xi F_{2\gamma}(\xi) = \frac{8i}{\sqrt{3}}\gamma\,\xi^2 K_{1/3}(\tfrac{4}{3}\gamma\,\xi^3)\ .$$

Integrating the latter between the limits ξ and ∞ we get

$$\int_{-\infty}^{\infty} e^{2i\gamma(\xi^2 x + \frac{1}{3}x^3)}\, x^{-1}\, dx = -\frac{2i}{\sqrt{3}} \int_{\frac{4}{3}\gamma\xi^3}^{\infty} K_{\frac{1}{3}}(\eta)\, d\eta \ .$$

Finally, on substitution of these results and making use of the recurrence relation

$$-2\, K_{\frac{2}{3}}'(x) = K_{\frac{5}{3}}(x) + K_{\frac{1}{3}}(x) \ ,$$

we get

$$\left.\begin{aligned} \overline{P_n^{(1)}} &= \frac{\sqrt{3}\mu_v e^2 c\, \omega_B^2 \sin\alpha}{8\pi^2 \xi}\ F^{(1)}(n/n_c) \ , \\ \overline{P_n^{(2)}} &= \frac{\sqrt{3}\mu_v e^2 c\, \omega_B^2 \sin\alpha}{8\pi^2 \xi}\ F^{(2)}(n/n_c) \ , \end{aligned}\right\} \qquad (12)$$

where

$$\left.\begin{aligned} F^{(1)}(x) &= \frac{1}{2}x \left\{ \int_x^{\infty} K_{\frac{5}{3}}(\eta)\, d\eta - K_{\frac{2}{3}}(x) \right\} , \\ F^{(2)}(x) &= \frac{1}{2}x \left\{ \int_x^{\infty} K_{\frac{5}{3}}(\eta)\, d\eta + K_{\frac{2}{3}}(x) \right\} , \end{aligned}\right\} \qquad (13)$$

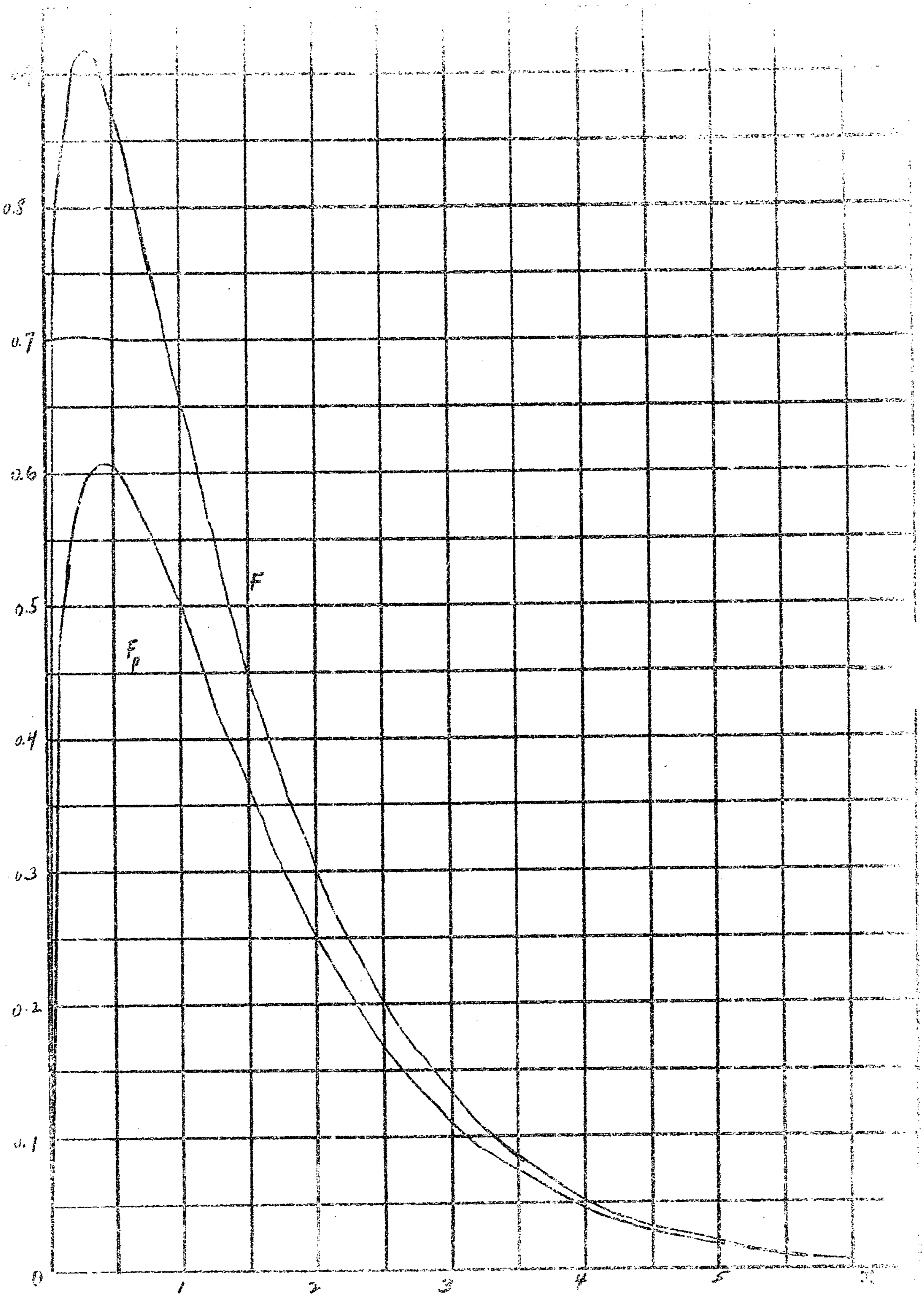
and

$$n_c = 3 \sin\alpha \,/\, 2\xi^3 . \qquad (14)$$

Since they believed that all the radiation emitted belonged to $\overline{P_n^{(2)}}$, Oort and Walraven gave a graph and table of values of the function

$$\begin{aligned} F(x) &= F^{(1)}(x) + F^{(2)}(x) \\ &= x \int_x^{\infty} K_{\frac{5}{3}}(\eta)\, d\eta \ , \end{aligned} \qquad (15)$$

only. For large x we have the asymptotic formula



The functions $F(x)$ and $F_p(x)$

Figure 3

$$F(x) \sim \sqrt{\left(\frac{\pi}{2}\right)}\, e^{-x}\, x^{1/2}\, \left(1 + \frac{55}{72x} - \frac{10151}{10368x^2}\right) \qquad (15')$$

It is represented in Table 2 and Figure 3, together with the complementary function

$$F_p(x) = F^{(2)}(x) - F^{(1)}(x) \qquad (16)$$
$$= x\, K_{2/3}(x),$$

whose asymptotic form is given by

$$F_p(x) \sim \sqrt{\left(\frac{\pi}{2}\right)}\, e^{-x}\, x^{1/2} \left(1 + \frac{7}{72x} - \frac{455}{10368x^2}\right). \qquad (16')$$

The values of $F(x)$ up to $x = 5.0$ are taken from Oort and Walraven's paper (save that the value for $x = 4$ has been altered from .0522).

TABLE 2

$$F(x) = x \int_x^{\infty} K_{5/3}(\eta)\, d\eta$$

$$F_p(x) = x\, K_{2/3}(x)$$

x	F	F_p	x	F	F_p
0	0	0	1.0	0.655	0.494
0.001	0.213	0.107	1.2	0.566	0.439
0.005	0.358	0.184	1.4	0.486	0.385
0.010	0.445	0.231	1.6	0.414	0.336
0.025	0.583	0.312	1.8	0.354	0.290
0.050	0.702	0.388	2.0	0.301	0.250
0.075	0.772	0.438	2.5	0.200	0.168
0.10	0.818	0.475	3.0	0.130	0.111
0.15	0.874	0.527	3.5	0.0845	0.0726
0.20	0.904	0.560	4.0	0.0541	0.0470
0.25	0.917	0.582	4.5	0.0339	0.0298
0.30	0.919	0.596	5.0	0.0214	0.0192
0.40	0.901	0.607	6.0	0.0085	0.0077
0.50	0.872	0.603	7.0	0.0033	0.0031
0.60	0.832	0.590	8.0	0.0013	0.0012
0.70	0.788	0.570	9.0	0.00050	0.00047
0.80	0.742	0.547	10.0	0.00019	0.00018
0.90	0.694	0.521			

Since $F(x)$ has its greatest values where $x = O(1)$ it follows that most of the radiation is emitted in harmonics $n \approx n_c$. The spectral character of the emission is thus dependent on the energy $\mathcal{E} = mc^2/\xi$ of the electron. In this range of n the harmonics are so closely spaced that the radiation is quasi-continuous. If we write $\overline{P_f}\, df$ for the mean power radiated in the band $(f, f + df)$ and

$$f = n\,\omega_B/2\pi \quad , \quad f_c = n_c\,\omega_B/2\pi \quad ,$$

we have

$$\overline{P_f}\, df = \overline{P_n}\, df/f_B \;,$$

so that

$$\overline{P_f} = \overline{P_f^{(1)}} + \overline{P_f^{(2)}} \;, \tag{17}$$

where

$$\left.\begin{aligned} \overline{P_f^{(1)}} &= \tfrac{1}{2}\sqrt{3}\mu_v e^2 c f_{BO} \sin\alpha\, F^{(1)}\left(\frac{f}{f_c}\right) , \\ \overline{P_f^{(2)}} &= \tfrac{1}{2}\sqrt{3}\mu_v e^2 c f_{BO} \sin\alpha\, F^{(2)}\left(\frac{f}{f_c}\right) , \end{aligned}\right\} \tag{18}$$

and

$$f_c = 3f_{BO}\sin\alpha/2\xi^2 \quad , \quad f_{BO} = eB_O/2\pi m \quad . \tag{19}$$

For B_O in gauss and a radiating electron,

$$\overline{P_f} = 2.34 \times 10^{-25}\, B_O \sin\alpha\, F\left(\frac{f}{f_c}\right) \text{ watt m}^{-2}\,(c/s)^{-1},$$

$$f_c = 4.20\, B_O \sin\alpha/\xi^2 \text{ Mc/s} \;, \quad f_{BO} = 2.80\, B_O \text{ Mc/s},$$

with appropriate suffixes where necessary. Thus, for $B_O = 10^{-4}$ gauss and $f_c = 100$ Mc/s we must have $\xi \cong 10^{-3}$.

Finally, as a check on the calculations made in this section, we should show that

$$\int_0^{\infty} \overline{P_f}\, df = P(t) \ . \qquad (20)$$

A comparison with Sec. 5.3(33) shows that this is equivalent to the mathematical result

$$\int_0^{\infty} F(x)dx = 8\pi/9\sqrt{3} \ ,$$

which may be verified by application of the result (see Section 5.5)

$$\int_0^{\infty} x^{s-1} \int_x^{\infty} K_{\nu+1}(y)dydx = \frac{\nu+s}{s}\int_0^{\infty} x^{s-1} K_\nu(x)dx, \qquad R\ell \ s > 0,$$

together with the formula

$$\int_0^{\infty} x^{s-1} K_\nu(x)dx = 2^{s-2}\Gamma(\tfrac{1}{2}s - \tfrac{1}{2}\nu)\Gamma(\tfrac{1}{2}s + \tfrac{1}{2}\nu), \qquad R\ell \ s > |R\ell \ \nu| \ . \qquad (21)$$

5.5 The Emission from a Distribution of Gyrating Electrons

At a given field point the synchrotron radiation received from a small volume containing a distribution of ultrarelativistically gyrating electrons effectively originates in the group whose velocities have directions within the conical annulus for which α is within an angular distance $O(\xi)$ of θ, the angle between $\underline{n}$ and $\underline{B}_0$. Each member of the group contributes the amounts $\overline{P_n^{(1)}}(\underline{n})$, $\overline{P_n^{(2)}}(\underline{n})$, which depend on both the energy

$$\mathcal{E} = \mathcal{E}_0/\xi \ , \qquad \mathcal{E}_0 = mc^2 \ , \qquad (1)$$

and the small angle $\psi = \alpha - \theta$. The contribution to the emissivity in the direction $\underset{\sim}{n}$ from members of the group in the same energy range $(\mathcal{E},\ \mathcal{E} + d\mathcal{E})$ is obtained by integrating the incoherent contributions over the range of α. In general, it may be assumed that the distribution of velocities is uniform with respect to α. Then it is clear that the integrations result in two linearly polarized contributions $\overline{P_n^{(1)}}/4\pi$, $\overline{P_n^{(2)}}/4\pi$, in which α is replaced by θ, in the orthogonal directions $\underset{\sim}{n}\wedge\underset{\sim}{i}$, $\underset{\sim}{i}$, respectively.

Let $N(\mathcal{E}/\mathcal{E}_0)d\mathcal{E}/\mathcal{E}_0$ be the local number density of electrons having energies within the range $(\mathcal{E},\ \mathcal{E} + d\mathcal{E})$. Then the monochromatic emissivity η_f consists of the two oppositely polarized components $\eta_f^{(1)}$, $\eta_f^{(2)}$ of the form

$$\eta_f^{(1)}\ (\underset{\sim}{n}) = \frac{1}{4\pi}\int_0^{\infty} N(\mathcal{E}/\mathcal{E}_0)\overline{P_f^{(1)}}\ (f/f_c)d\mathcal{E}/\mathcal{E}_0\ , \text{ etc.}, \qquad (2)$$

where, by (1) and Sec. 5.4(19),

$$f_c = \tfrac{3}{2}f_{BO} \sin\theta\ (\mathcal{E}/\mathcal{E}_0)^2\ . \qquad (3)$$

Substitution from Sec. 5.4(18) with an appropriate change of variable then yields the result

$$\eta_f^{(1)}\ (\underset{\sim}{n}) = \frac{\mu_v e^2 c}{8\sqrt{2\pi}}\ (f f_{BO} \sin\theta)^{\frac{1}{2}}$$

$$\times \int_0^{\infty} N\left\{\left(\frac{2f}{3f_{BO}\sin\theta}\right)^{\frac{1}{2}} x^{-\frac{1}{2}}\right\} x^{-\frac{3}{2}}\ F^{(1)}\ (x)dx\ , \qquad (4)$$

etc.

The total emissivity is given by

$$\eta_f = \eta_f^{(1)} + \eta_f^{(2)}\ , \qquad (5)$$

of which the part

$$\eta_f^{(p)} = \eta_f^{(2)} - \eta_f^{(1)} \tag{6}$$

is polarized in the direction of $\underset{\sim}{i}$, which is perpendicular to the projection of $\underset{\sim}{B}_0$ on the plane transverse to $\underset{\sim}{n}$. By Sec. 5.4(15) and (16), η_f and $\eta_f^{(p)}$ are given by formulae like (4), with $F(x)$ and $F_p(x)$ respectively in the integrand. The degree of polarization of the radiation emitted is then

$$p_f = \eta_f^{(p)}/\eta_f \, . \tag{7}$$

These results are simplified if, as is frequently the case, the energy spectrum of the emitting electrons can be represented by a power law with cut-offs at energies $\mathcal{E}_1$, $\mathcal{E}_2$. Then

$$N(x) = \left.\begin{array}{ll} A\, x^{-\gamma}, & \mathcal{E}_1/\mathcal{E}_0 \leqslant x \leqslant \mathcal{E}_2/\mathcal{E}_0 \, , \\ 0, & x < \mathcal{E}_1/\mathcal{E}_0,\ x > \mathcal{E}_2/\mathcal{E}_0 \, , \end{array}\right\} \tag{8}$$

where A is proportional to the local number density of all the electrons and the index γ is a constant. We then have for the emissivity in a direction making an angle θ with the magnetic field $\underset{\sim}{B}_0$ in which electrons are gyrating,

$$\eta_f(\underset{\sim}{n}) = A\,\frac{\mu_v e^2 c}{8\sqrt{2\pi}}\left(\frac{3}{2}\right)^{\gamma/2} (f_{B0}\sin\theta)^{(\gamma+1)/2}\, f^{-(\gamma-1)/2}\left\{G\left(\frac{f}{f_{c2}}\right) - G\left(\frac{f}{f_{c1}}\right)\right\},$$

where

$$G(x) = \int_x^{\infty} \xi^{(\gamma-3)/2}\, F(\xi)\, d\xi \quad , \tag{10}$$

and a similar pair of formulae for $\eta_f^{(p)}$ and G_p. Both functions depend on the frequency of the radiation and the component of the magnetic field transverse to the direction of emission. The dependence is that of a simple power law when ξ_1 and ξ_2 are such that $f/f_{c2} \ll 1$ and $f/f_{c1} \gg 1$ in the range of frequencies of interest. Then the degree of polarization is a constant independent of the frequency and the magnetic field. For the Crab Nebula, Woltjer (1958) finds that the radio-frequency spectrum is indeed given by a power law of index about -0.35, which corresponds to the value $\gamma = 1.7$. For radio frequencies therefore, the expression in braces in (9) becomes equal to the constant G(0). Clearly, any departure from a power-law spectrum at higher frequencies can be attributed to the circumstance that the values of the argument f/f_{c2} have now become significant. In fact it is found that the magnitude of the spectral index increases for wavelengths below 3000 A.

In order to find expressions for η_f, $\eta_f^{(p)}$ we therefore need to evaluate the functions

$$G(x) = \int_x^\infty \xi^{(\gamma-1)/2} \int_\xi^\infty K_{5/3}(\eta)\, d\eta\, d\xi \quad , \tag{11}$$

$$G_p(x) = \int_x^\infty \xi^{(\gamma-1)/2} K_{2/3}(\xi)\, d\xi \quad . \tag{12}$$

It can be shown that

$$\int_x^\infty \xi^{s-1} \int_\xi^\infty K_{\nu+1}(\eta)\, d\eta\, d\xi = \frac{\nu+s}{s} \int_x^\infty \xi^{s-1} K_\nu(\xi)\, d\xi$$

$$- \frac{x^s}{s}\left\{\int_x^\infty K_{\nu+1}(\xi)\, d\xi - K_\nu(x)\right\} ,$$

whence

$$G(x) = \frac{\gamma + 7/3}{\gamma + 1} G_p(x) - \frac{2x^{(\gamma-1)/2}}{\gamma + 1}\left\{F(x) - F_p(x)\right\} . \tag{13}$$

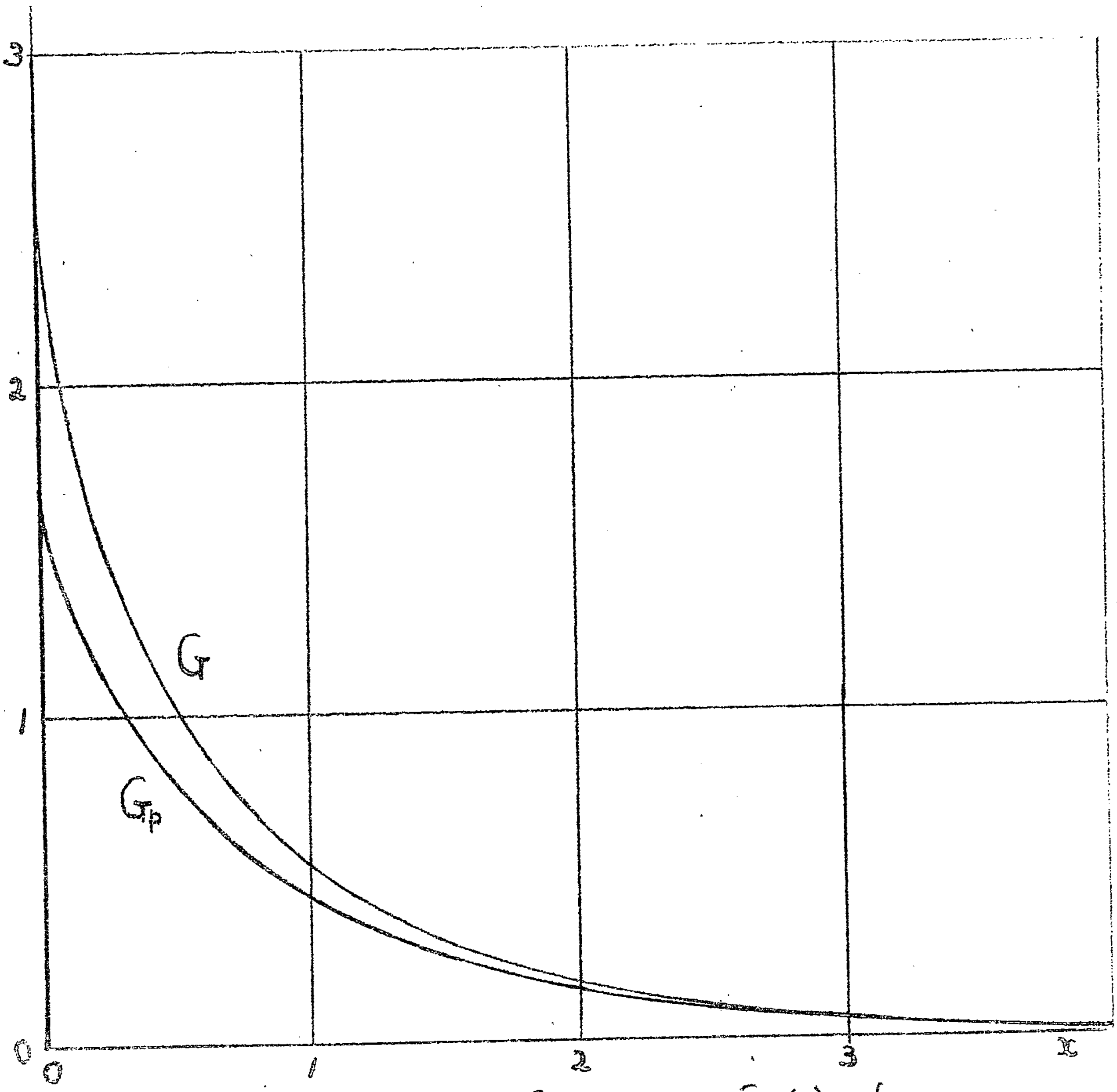


Figure 4. The functions $G_T(x)$ and $G_{Tp}(x)$ for $\gamma = 5/3$

To face p. 139

Thus, the emissivity and degree of polarization can be evaluated in terms of the tabulated functions $F(x)$, $F_p(x)$ and the function $G_p(x)$ given by (12). The value $G(0)$ is given by Sec. 5.4(21), viz.,

$$G_p(0) = 2^{(\gamma-3)/2}\,\Gamma\left(\frac{3\gamma-1}{12}\right)\Gamma\left(\frac{3\gamma+7}{12}\right) , \quad \gamma > 1/3 . \tag{14}$$

For other values of x the function cannot in general be evaluated in terms of tabulated functions.

There is, however, an exception in the particular case where $\gamma = 5/3$. Then we may apply the formula

$$\int_x^\infty \xi^{1-\nu} K_\nu(\xi)\,d\xi = x^{1-\nu} K_{\nu-1}(x) , \quad \mathcal{R}\ell\ \nu < 1 ,$$

to get

$$G_p(x) = x^{1/3} K_{1/3}(x) , \tag{15}$$

whose asymptotic form is given by

$$G_p(x) \approx \sqrt{\left(\frac{\pi}{2}\right)}\, e^{-x}\, x^{-1/6} \left(1 - \frac{5}{72x} + \frac{385}{10368x^2}\right) . \tag{15'}$$

It is fortuitous that this value of γ is applicable to observations of the Crab Nebula. Then

$$G(x) = \frac{3}{2} G_p(x) - \frac{3}{4} x^{1/3} \left\{ F(x) - F_p(x) \right\} , \tag{16}$$

whose asymptotic form is given by

$$G(x) \approx \sqrt{\left(\frac{\pi}{2}\right)}\, e^{-x}\, x^{-1/6} \left(1 + \frac{43}{72x} - \frac{17375}{10368x^2}\right) . \tag{16'}$$

The functions $G(x)$, $G_p(x)$ for $\gamma = \frac{5}{3}$ are represented in Table 3 and Figure 4. The same general forms may be expected for neighboring

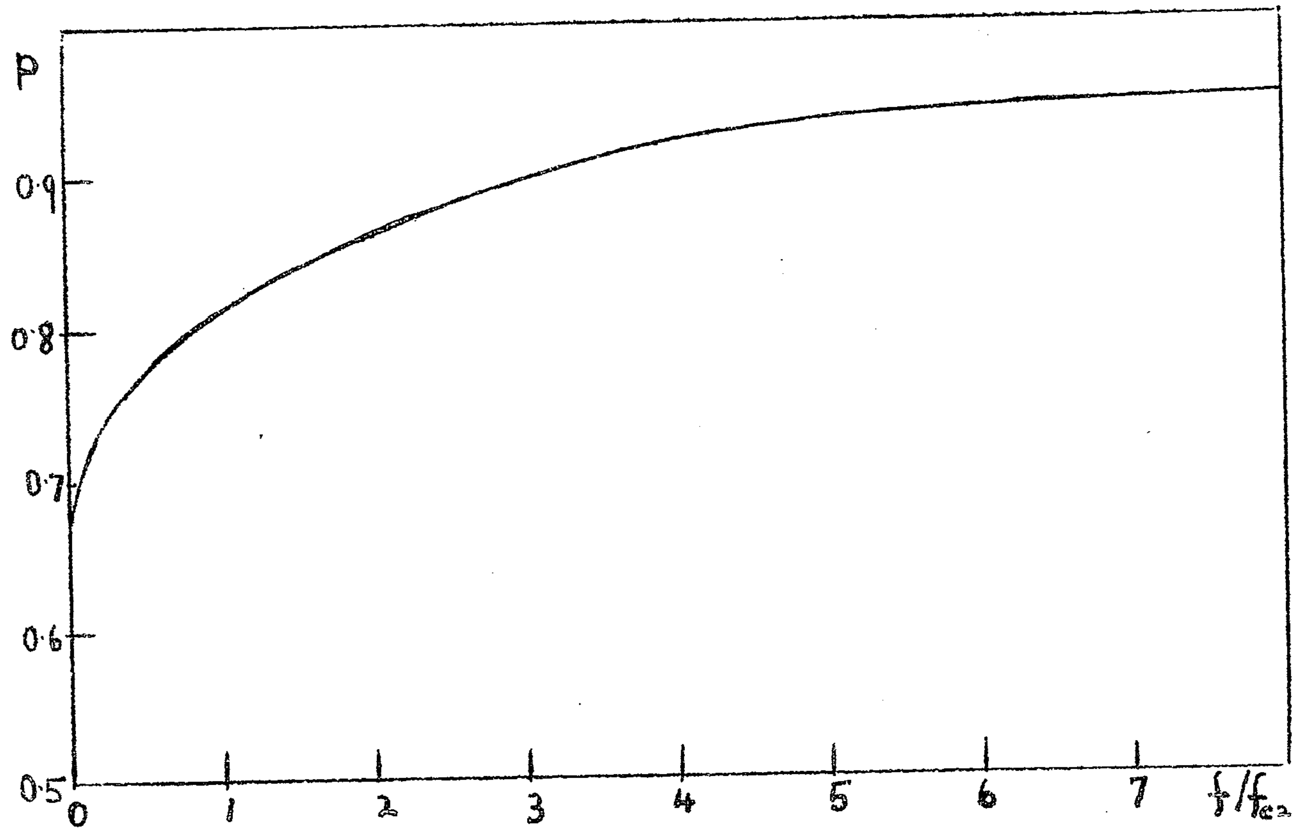


Figure 5. The degree of polarization of the emissivity for $\gamma = 5/3$

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values of γ.

TABLE 3

$$G(x) = \frac{3}{2}\, G_p(x) - \frac{3}{4}\, x^{1/3}\left\{F(x) - F_p(x)\right\}$$

$$G_p(x) = x^{4/3} K_{1/3}(x)$$

x	G	G_p	x	G	G_p
0	2.531	1.688	2.0	0.172	0.147
0.2	1.585	1.158	2.5	0.097	0.086
0.4	1.170	0.888	3.0	0.056	0.051
0.6	0.891	0.696	4.0	0.019	0.018
0.8	0.690	0.551	5.0	0.0068	0.0064
1.0	0.537	0.438	6.0	0.0024	0.0023
1.2	0.425	0.351	7.0	0.00087	0.00082
1.4	0.338	0.281	8.0	0.00031	0.00029
1.6	0.271	0.226	9.0	0.000112	0.000106
1.8	0.215	0.182	10.0	0.000040	0.000038

Then the emissivity is given by the formula

$$\eta_f(\underset{\sim}{n}) = A\, \frac{\mu_v e^2 c}{8\sqrt{2\pi}} \left(\frac{3}{2}\right)^{5/6} \left(f_{B0} \sin\, \theta\right)^{4/3} f^{-1/3}\, G\left(\frac{f}{f_{c2}}\right) \tag{17}$$

and the corresponding degree of polarization by

$$p_f(\underset{\sim}{n}) = G_p\left(\frac{f}{f_{c2}}\right) \Big/ G\left(\frac{f}{f_{c2}}\right) , \tag{18}$$

where, by (3),

$$\frac{f}{f_{c2}} = \frac{2f}{3f_{B0} \sin\, \theta} \left(\frac{\mathcal{E}_0}{\mathcal{E}_2}\right)^2 . \tag{19}$$

The functional form of (18) is represented in Figure 5. For a given direction the degree of polarization of the radiation emitted increases steadily with frequency, from the value $\frac{2}{3}$ asymptotically to the value 1.

The observed intensity of radiation emitted from a distribution of gyrating electrons is determined by the equation of transfer along a ray trajectory. Such a medium is usually so tenuous that the refractive index is unity and absorption* is negligible. Thus, the trajectories are rectilinear and the intensity is simply given by the integral along a trajectory

$$I_f = \int \eta_f ds \quad . \tag{20}$$

Everywhere along the trajectory there are contributions $\eta_f^{(1)}$, $\eta_f^{(2)}$ polarized in the transverse plane, along the projection of the direction of $\underset{\sim}{B}_0$ and in the perpendicular direction, respectively. As long as the orientation of $\underset{\sim}{B}_0$ remains unchanged the intensity will consist of similarly polarized components $I_f^{(1)}$, I_f given by integrals of the form (20). In particular, if A and the magnitude of $\underset{\sim}{B}_0$ also remain constant, the degree of polarization of the ray will be given by (18). The direction of polarization measured optically over the Crab Nebula (Woltjer 1957) varies systematically, which indicates a definite structure in the magnetic field distribution. However, the degree of polarization varies from values below 50 percent in the denser central regions to as high as 80 percent at the edges.

In an actual astronomical situation the rays emerging from an object may have traversed regions in which the magnetic field varies considerably in both magnitude and direction. This is more likely in rays from the center than in rays from the edges of the

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* Given the spontaneous emission from an electron (Sec. 5.4), the absorption coefficient may be calculated from the Einstein coefficients (cf. Twiss, 1954).

object. The overall effect on an emergent ray is one of depolarization. Its resultant state of polarization is conveniently calculated by defining an emissivity polarization tensor which specifies the amount and state of polarization of the radiation emitted at each point of the trajectory. The integral of this tensor will yield an intensity polarization tensor (Westfold 1958) whose two characteristic values are the intensities of the two oppositely polarized components $I_f^{(\alpha)}$, $I_f^{(\beta)}$ into which it is possible to resolve an arbitrary superposition of incoherent partially polarized rays; the characteristic vectors $\underset{\sim}{i}_\alpha$, $\underset{\sim}{i}_\beta$ will be in the directions of polarization of the two components. Alternatively, the partially polarized emergent ray may be specified by its total monochromatic intensity

$$I_f = I_f^{(\alpha)} + I_f^{(\beta)} \quad , \tag{21}$$

and the intensity of the excess component

$$I_f^{(p)} = I_f^{(\alpha)} - I_f^{(\beta)} , \tag{22}$$

polarized in the direction of $\underset{\sim}{i}_\alpha$. Its degree of polarization is therefore

$$p_f = I_f^{(p)}/I_f \quad . \tag{23}$$

The measured degree of polarization will not be given exactly by (23), for the measuring instrument is limited by both its resolving power and its frequency bandwidth. The former effect is determined by the diffraction pattern of the instrument, which in the radio case may be different for different polarizations. With the appropriate pattern, rays of each polarization make their contributions in accordance with the relations of Sec. 1.4 or their optical equivalents. This depolarizing effect probably accounts for Westerhout's (1956) failure to detect more than 1 percent of polarization at 22 cm with an aerial of 2°.5 beamwidth. The diameter of the Crab Nebula is about 5'. The latter effect will involve a similar averaging over the receiver bandwidth. It will be insignificant unless the rays suffer Faraday rotation in a magneto-ionic medium between the source and the observing instrument. Then each spectral component suffers a different rotation so that the overall effect is again one of depolarization. This will be discussed in a forthcoming paper. The situation is still more complicated if the emitting region is sufficiently dense for the Faraday effect to be appreciable there.

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EXAMPLES

1. The distribution of brightness temperature of 100 Mc/s rays from a circular-disk source subtending an angle of 3° is given by

$$T_b = 5000\ (1 + \cos 2\pi\theta/3), \quad 0 \leqslant \theta \leqslant 3/2,$$
$$0, \qquad \theta \geqslant 3/2,$$

where θ° is the angular distance of a ray from the center of the disk. Find the apparent temperature of the source and its flux density (one polarization) in m.k.s. units.
(It is advisable to carry out mathematical operations with angles expressed in radian measure.)

2. The receiving properties of an aerial are specified approximately by the gain function

$$G(\theta,\phi) \quad = \quad \frac{\sin^2 20\theta}{2\theta^2} \qquad (\theta \text{ in radians}).$$

Plot G against angular distance in degrees and read off the distance between the half-power points. Indicate how you would calculate the effective solid angle of the aerial and how you would compare its angular width with the distance between the half-power points.
(The location of the maxima and values of the function $\sin^2 x/x^2$ are given in books on physical optics where diffraction by a slit is treated.)

3. On a 100 Mc/s equipment of bandwidth 1 Mc/s and noise factor 6 (ambient temperature 300°K) a source of flux density 3.5×10^{-25} watts m^{-2} $(c/s)^{-1}$ can just be detected against a background aerial temperature of 600°K. By what factor should the overall time constant of the equipment be increased in order to detect a source of strength 5.0×10^{-25} watts m^{-2} $(c/s)^{-1}$ against a background aerial temperature of 3000°K?

4. Given the formula for the available power from a point source, prove that in a twin-aerial interferometer the available power in the bandwidth Δf from a small uniform source of angular width 2W at an angle θ from the interferometer axis is

$$P_o \left(1 + \cos D \frac{\sin \alpha W}{\alpha W} \right) ,$$

where

$$D = \frac{2 \pi d}{\lambda} \sin \theta, \qquad \alpha = \frac{2 \pi d}{\lambda} \cos \theta , \qquad \text{and}$$

P_o is the power that would be available from a single aerial. Show that if $\lambda \ll d \cos \theta$ the angular distance between successive maxima of cos D is $2\pi/\alpha$ and explain why this will represent the fringe spacing of the instrument. Derive a condition on the width of the source that will ensure that the maxima of available power remain sensibly located at the fringe maxima.

5. Prove that in the above instrument the effect of the selection of a finite bandwidth of the power from a point source is represented by the formula

$$P = P_o \quad 1 + \cos D \frac{\sin (D\Delta f/2f)}{(D\Delta f/2f)} \quad .$$

Again, derive a condition on the bandwidth that will ensure that the maxima of available power remain at the fringe maxima, where $D = 2n\pi$. Plot the ratio $(P_{max} - P_{min})/2P_o$ against the fringe number n for values of n from 0 to 50 and $\Delta f/f = 0.01, 0.02, 0.04$.

6. Given Snell's law in the usual form, show that for conditions of spherical symmetry a ray trajectory is determined by the equation $\mu\ r \sin \phi = d$, where ϕ is the angle between the radius vector and the tangent at the distance r from the center. If $\mu = 1$ at $r = \infty$ interpret the constant d. Obtain the equation of the half-trajectory in polar coordinates

$$\theta - \theta_o = d \int_r^{\infty} \frac{dr}{r\sqrt{(\mu^2 r^2 - d^2)}} \quad ,$$

How is the value of r at the turning point of this trajectory determined?

7. A plane monochromatic electromagnetic wave is traveling in a uniform ionized medium in a direction parallel to and in the same sense as the imposed magnetic field. From first principles show that the complex refractive index q is given by

$$q^2 = K_1 = 1 - \frac{x}{1 + iz - y} \quad ,$$

$$\text{or} \quad K_2 = 1 - \frac{x}{1 + iz + y} \quad .$$

Show also that the wave is circularly polarized, and determine the sense of polarization in either case. Assuming that $z \ll 1$, find the ratio of the two absorption coefficients for relatively uninhibited propagation.

8. Assuming Snell's law, prove that for a bundle of ray trajectories refracted at the boundary between two media

$$\mu_1^2 dS_1 d\Omega_1 = \mu_2^2 dS_2 d\Omega_2 .$$

9. Derive Bottlinger's formula

$$V_g = R_0 \left\{ \omega(R) - \omega_0 \right\} \sin \ell'$$

for the radial component of the velocity of a point P relative to the point P_0, when both move with negative angular velocities $\omega(R)$, $\omega_0 = \omega(R_0)$ about a fixed center O and ℓ' is the angle PP_0O

10. From a certain direction continuous radiation of brightness temperature T_c together with line-radiation from a group of HI clouds at kinetic temperature T is incident on a receiving antenna. Prove that the brightness temperature of the incident radiation is given b

$$T_b = T_c + (Te^{-\tau_0} - T_c')(1-e^{-\tau}) ,$$

where τ is the optical thickness of the cloud group, τ_0 the optical thickness of the trajectory between the group and the antenna, and T_c' that part of T_c due to sources beyond the group.

RADIO ASTRONOMY

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