Поздравляю с новым годом. Большое спасибо за гостеприимство, а также за научные дискуссии.
А. Соколов
24.XII.63г.

Dr. K. Kellermann
CSIRO Radiophysics Lab. Sydney N.S.W.
Australia.

SYNCHROTRON RADIATION

Lectures delivered at

THE AUSTRALIAN NATIONAL UNIVERSITY

December 2-21, 1963

by

PROFESSOR A.A. SOKOLOV

Moscow State University.

PREAMBLE

First of all I would like to transmit to the staff of the Australian National University warm greetings from their colleagues in the Moscow State University. I would also like to express the hope that the cultural exchange programme which is being initiated will not decay, but grow in accordance with an exponential law. Our two universities have everything to gain from this programme.

Personal acquaintance and face-to-face contacts between scientists are incomparably better than acquaintance through literature. Such contacts will enable us to familiarize ourselves with the best aspects of each of our universities, to find common points of view and to achieve mutual understanding. I also hope that we shall be able to assist one another in the training of young scientists.

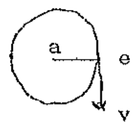
Scientific discussions which will arise among us, no matter how sharply we may happen to disagree, will stimulate each of us and prompt us to review once again our scientific theories and to try to perfect our theoretical investigations.

In the Moscow State University I conduct a course on quantum mechanics for fourth-year students. During my visit to Canberra I would like to devote my lectures to a comparatively narrow topic, the THEORY OF SYNCHROTRON RADIATION. Together with Professor Ivanenko I began to study this problem in 1945 and since then it continues to attract me and I often return to it.

I. INTRODUCTION AND CLASSICAL THEORY

1. Statement of problem

Consider particle of charge e (electron) rotating in circle of radius a with constant tangential velocity v.



Contrary to first appearances this problem is not simple, and a complete solution is not yet available, especially for ultra-relativistic energies. Recent experimental data for this case have impelled further study of the theory.

Classical situation $v \ll c$

$$\omega = \frac{v}{a} \text{ is angular velocity}$$

$$\lambda \sim \frac{c}{\omega} = \left(\frac{c}{v}\right) a \gg a \text{ is wavelength of radiation.}$$

Ultra-relativistic case ($v \rightarrow c$) has several differences from classical :

i) maximum radiation occurs at very high harmonics, harmonic number
$$\nu \sim \left(\frac{E}{m_o c^2}\right)^3$$
For electron $m_o c^2 = 0.5$ MeV; if total energy is $E = 50$ MeV, then $\nu \sim 10^6$, so $\lambda \sim 10^{-6}$ a. Thus even for macroscopic orbits, synchrotron radiation will still be in visible or even ultra-violet range.

ii) Radiation is not constant but fluctuating; a finite number of quanta are emitted per revolution. Thus the quantum nature of radiation appears here on a macroscopic scale, for electrons at $E \gtrsim 500$ MeV. This is not just a small perturbation but a major effect.

iii) Radiation is highly polarized, affected by spin properties of electron; thus one can study both features.

2. Experimental possibilities for electron synchrotrons

i) Photoproduction of particles

ii) Electromagnetic structures of proton and neutron

Now up to $\sim$ 1 Bev (Hofstadter and Wilson)

Soon at $\sim$ 6 Bev (Livingston's machine)

Under construction for $\sim$ 7 Bev (Hamburg, Erevan)

iii) Storage rings : ADA (Frascati) at 200 MeV

Under construction for 1500 MeV

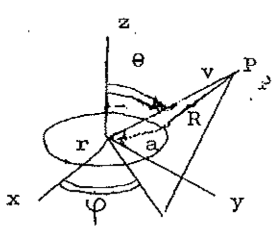
Budker (Novosibirsk) at 100 MeV

Under design (Livingston) for 3 Bev

Classically there would be no radiation from a continuous ring of current, but quantum mechanically it cannot be avoided. This energy loss is made up from external sources (magnetic field). Storage rings make possible the preservation of anti-matter : positrons live $\sim$48 hours in ADA.

3. Classical theory of electron radiation

i) Let the electron orbit of radius a be in this xy-plane centred at the origin. We observe the radiation at a point $P = (r, \theta, \varphi)$ in polar coordinates. The vector potential at P is given by the Lienard-Wiechert form :



$$\overline{A} = \left(\frac{e}{c}\right) \int \frac{\overline{v}(\tau)}{R} \, \delta\left(\tau - t + \frac{R}{c}\right) d\tau \qquad \ldots(1)$$

where the time coordinate is τ at the electron and t at the observation point P, while R is the instantaneous distance from the electron to P, given by

$$R = r\left(1 - \frac{2\,\underset{\sim}{r}\,\underset{\sim}{a}}{r^2} + \frac{a^2}{r^2}\right)^{\frac{1}{2}} \approx r - a \sin\theta \cos\chi \qquad \ldots(2)$$

where $\chi = \omega t - \varphi$ is the angle of the electron in its plane, relative to the projection of P.

<u>Note</u>: Approximation (2) ignores terms of order $\frac{a^2}{r^2}$; classical theory cannot do better, and this approximation underlies all the radiation formulae of this lecture.

ii) In formula (1) the δ-function has the expansion

$$\delta(\tau') = \frac{1}{T} \sum_{\nu=-\infty}^{\infty} e^{i\nu\left(\frac{2\pi\tau'}{T}\right)} \qquad \ldots.(3)$$

$$= \frac{\omega}{2\pi} \sum_{\nu=-\infty}^{\infty} e^{i\nu\omega\tau'}$$

Therefore

$$\overline{A} = \sum_{\nu=\infty}^{\infty} \overline{A}(\nu)\, e^{-i\nu\gamma}, \quad \gamma = \omega t - \omega R/_c - \varphi + \pi/_2 \qquad \ldots.(4)$$

From this we can compute E_θ, E_φ and H_θ, H_φ and so compute the radial Poynting vector at point P, which is the outward flux of radiated energy

$$\overrightarrow{\gamma_r} = \left(\frac{c}{4\pi r}\right) \left(E_\theta H_\varphi - H_\theta E_\varphi\right) \qquad \ldots.(5)$$

We are interested in the time-averaged flux at point P, for which we use the fact that

$$\frac{1}{T}\int_0^T \cos\nu\left(\frac{2\pi t}{T}\right) \cos\nu'\left(\frac{2\pi t}{T}\right) dt = \delta_{\nu\nu'} \ldots.(6)$$

iii) Suppressing further details, we obtain the average energy flux per unit solid angle at P ,

$$\left(\frac{dW}{d\Omega}\right) = \frac{1}{T}\int_0^T \gamma_r\, dt = \sum_{\nu=1}^{\infty} \frac{dW_\nu}{d\Omega} \qquad \ldots.(7)$$

$$\left(\frac{dW_\nu}{d\Omega}\right) = \frac{e^2\nu^2\beta^4 c}{2\pi\ a^2} \left[\cot^2\theta\ J_\nu^2(\nu\beta\sin\theta) + \beta^2 J_\nu'^{\,2}(\nu\beta\sin\theta)\right]$$

where J_ν = Bessel function, J_ν' its derivative.

The sum over ν can be made in closed form :

$$\frac{dW}{d\Omega} = \frac{e^2\beta^4 c}{8\pi\ a^2} \frac{\left[1 + \cos^2\theta - \frac{1}{4}(1+3\beta^2)\beta^2 \sin^4\theta)\right]}{(1-\beta^2\sin^2\theta)^{7/2}}\, d\Omega \qquad \ldots.(8)$$

As $\beta \longrightarrow 1$, radiation all concentrates in plane of electron, with beam width of order $\Delta\ \theta \sim \sqrt{1-\beta^2} = m_o c^2/_E$.

The angular integral yields

$$W_\nu = \int \left(\frac{dW_\nu}{d\Omega}\right) d\Omega = \frac{e^2 \nu c}{a^2} \left[2\beta^2 J'_{2\nu}(2\nu\beta) - (1-\beta^2) \int_o^{2\nu\beta} J_{2\nu}(X)\ dX \right] \qquad \ldots.(9)$$

Integration of Eq.(8) or summation of Eq.(9) yields the total radiation rate

$$W = \frac{2}{3} \quad \frac{e^2 \beta^4 c}{a^2} \quad (E/_{m_o c^2})^4 \qquad \ldots.(10)$$

iv) Historical remarks: These closed forms were unknown at the beginning of study of synchrotron radiation. They were afterwards found to have been published by Shott in 1912, who was trying (unsuccessfully) to obtain a classical theory for the radiation of atoms.

v) Asymptotic forms. Formula (9) is not very transparent for showing which harmonics contain the most energy. For this we need asymptotic approximations to the Bessel functions that are accurate near the maximum of the functions. By methods akin to the WKB approximation it is found that

$$J_\nu(X) \sim \frac{1}{\pi} \sqrt{2/_3(1-X/_\nu)} \quad K_{1/_3}\left[\nu/3 (2(1-X/_\nu))^{3/_2} \right] \ldots(11)$$

where K is a Bessel function of imaginary argument of the second kind.

Then

$$W_\nu d\nu \approx \frac{1}{\pi\sqrt{3}} \quad \frac{e^2 c}{a^2} \left(\frac{m_o c^2}{E}\right)^2 \nu d\nu \int_{\nu/\nu_o}^{\infty} K_{5/_3}(X)\ dX \qquad \ldots(12)$$

where

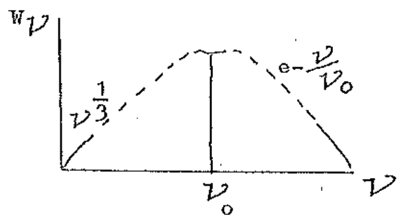
$$\nu_o = \frac{3}{2} (E/_{m_o c^2})^3 \qquad \ldots(13)$$

This formula was obtained by Ivanenko and Sokolov (1948) and by Schwinger (1949). An indirect check is obtained by using it to calculate the total energy radiated:

$$\int_0^\infty W_\nu \, d\nu = \frac{2}{3} \left(\frac{e^2 c}{a^2}\right) \left(E/m_o c^2\right)^4 \qquad \ldots(14)$$

This differs from the exact formula (10) only by the absence of a factor β^4, suggesting that the approximation (11) is valid so long as $\beta \approx 1$, which is just the ultra-relativistic case under consideration.

Graphically



$$\nu_o = \frac{3}{2}\left(E/m_o c^2\right)^3$$

vi) Experimental verification. Pollock et al. (1948) saw bluish light from a 70 MeV electron synchrotron, tangential to the electron orbit. In 1956, Cherenkov et al. studied specially at Physical Institute of Academy of Sciences in Moscow; also studies by Tamboulian et al. (1956).

These all show good agreement with theory. In spite of this, one finds that quantum effects are unexpectedly important for electron energies $\gtrsim$ 500 MeV, and that one cannot proceed without considering them. They will be the subject of the next lecture.

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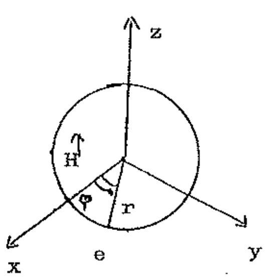
II. QUANTUM EFFECTS IN SYNCHROTRON RADIATION

1. Introduction

It may seem strange at first sight to consider quantum theory for electrons circulating with a macroscopic radius. But the results of these considerations have wide practical use.

We give here a strict quantum-mechanical solution, although with some algebraic approximations. This problem was first undertaken by the speaker and a student in 1953; later it was considered by other Russian, German, and American theorists. Only recently has a complete, detailed solution been available, which is summarized in the following.

2. Setting up of problem



Cylindrical coordinates

$\varphi, \quad r = \sqrt{x^2 + y^2}, \quad z$

The electron rotates in the xy-plane in an orbit with radius r and angular coordinate φ.
A constant (in time) magnetic field is applied externally, such that in the plane of the orbit

$$H_x = H_y = 0 \qquad H_z = b\, r^{-q}$$

A suitable vector potential for such a field is

$$A_z = 0\ , \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix} \left[1/(2-q) + q\, z^2/r^2 \right] b\, r^{-q} \qquad \ldots(1)$$

As the equation of motion to be solved, we take the relativistic scalar (Klein-Gordon) equation. The problem can also be done for the Dirac equation, in which case one sees the effects of electron spin. The Klein-Gordon equation for electron energy E is

$$(E^2 - c^2 P^2 - m_o^2 c^4) \Psi = 0$$

$$\overrightarrow{P} = \hbar/_i \overrightarrow{\nabla} - e/_c \overrightarrow{A} \qquad \ldots(2)$$

By the method of separation of variables, the solution has the form

$$\Psi = \sqrt{\frac{m_o c^2}{E}} \left(\frac{e^{i \ell \varphi}}{\sqrt{2\pi}}\right) \frac{u(r)}{\sqrt{r}} v(z) \qquad \ldots(3)$$

$\ell = 0, 1, 2, \ldots$ = azimuthal quantum number.

The functions u and v satisfy differential equations of the form

$$\frac{d^2u}{dr^2} + f(r, \ell) u = 0 \qquad \ldots (4a)$$

$$\frac{d^2v}{dz^2} + F(z^2, r, \ell) v = 0 \qquad \ldots(4b)$$

Equation 4(a) is solved under the <u>harmonic approximation</u> as follows: for each ℓ there exists an equilibrium point r = a at which $f'(a, \ell) = 0$. This point depends on the value of ℓ,

$$a(\ell) = \left\{\frac{\ell c \hbar (2-q)}{e b (1-q)}\right\}^{1/(2-q)} \qquad \ldots(5)$$

Introduce the coordinate $\rho = r - a(\ell)$, which characterizes the radial betatron oscillations about the equilibrium orbit $a(\ell)$, and expand $f(r,\ell)$ in terms of ρ :

$$f(r,\ell) = f(a,\ell) + \rho^2/_2 \, f''(a,\ell) + \ldots \qquad \ldots(6)$$

The anharmonic, higher terms in Eq.(6) are neglected. Then

$$\frac{d^2u}{d\rho^2} + (\alpha - \lambda^2 \rho^2)\, u = 0$$

$$\lambda = -\tfrac{1}{2} f''\,(a,\ell) = \frac{eH(a)}{c\hbar}\,(1-q)^{\frac{1}{2}} \qquad \ldots(7)$$

Here λ is positive because f" is intrinsically negative; and H(a) is the uniform magnetic field that acts to keep the electron in the orbit, so that

$$a\, e\, H(a) = \beta\, E \qquad \ldots(8)$$

Equation (7) is a standard form, for which the eigenvalues and corresponding eigenfunctions are

$$\alpha = \lambda\,(2s + 1) \quad s = 0,\ 1,\ 2,\ \ldots = \text{radial quantum number}$$

$$u_s = (\lambda/_\pi)^{\frac{1}{4}}\,(2^s s!)^{-\frac{1}{2}}\, e^{-\frac{1}{2}\lambda\rho^2}\, H_s(\sqrt{\lambda}\,\rho) \quad \ldots(9)$$

H_s = hermite polynomial.

Equation (4b) can also be solved in harmonic approximation, and one obtains a similar axial quantum number k = 0. 1. 2, ...

3. Quantum numbers

The quantum numbers of this system are ℓ,s,k, each running over the intergers 0, 1, 2, ... We seek to interpret these numbers.

The energy of the electron is $E = E(\ell,s,k)$. By Ehrenfest's theorem we can associate with each quantum number an angular frequency

$$\omega_\ell = \frac{1}{\hbar}\left(\frac{dE}{d\ell}\right) = \beta\,\omega_o = \left(\frac{v}{c}\right)\left(\frac{c}{a}\right)$$

$$\omega_s = \frac{1}{\hbar}\left(\frac{dE}{ds}\right) = \sqrt{1-q}\;\omega_\ell \qquad \ldots(10)$$

$$\omega_k = \frac{1}{\hbar}\left(\frac{dE}{dk}\right) = \sqrt{q}\;\omega_\ell$$

From this we can see that stable oscillations of the orbit are possible only for

$$0 < q < 1 \qquad \ldots(11)$$

Direct physical interpretations: ℓ characterizes radius of equilibrium orbit, Eq.(5); for s and k consider

$$\overline{\rho^2} = \int \rho^2 u_s^2 \, d\rho = \frac{\hbar c s}{e H(a) \sqrt{1-q}}$$

$$\ldots(12)$$

$$\overline{z^2} = \int z^2 v_k^2 \, dz = \frac{\hbar c k}{e H(a) \sqrt{q}}$$

Thus s and k specify the mean square amplitudes of radial and axial oscillations about the equilibrium orbit. These results could be obtained classically.

4. Radiation formula

As in the atomic case, the radiative transition rate $w_{nn'}$ depends on the quantum numbers of the initial state $n = (\ell s k)$ and of the final state $n' = (\ell' s' k')$. We now introduce the interaction of the electron with the emission field and solve the spontaneous radiation problem by assuming no photons present initially.

The radiation interaction can be written

$$V = -e \alpha_\mu \varphi_\mu \qquad \text{(Dirac equation)}$$

$$\ldots(13)$$

$$\alpha_\mu \longrightarrow P_\mu / m_0 c \qquad \text{(present case, P as in Eq.(2))}$$

Here

$$\varphi_\mu = \frac{1}{\sqrt{V}} \sum_{\kappa} \sqrt{\frac{2\pi \hbar c}{\kappa}} \; a_\mu^*(\vec{\kappa}) \; e^{i(c\kappa t - \vec{\kappa} \cdot \vec{r})} \ldots(14)$$

where V is the normalization volume of a large box; and when no photons are initially present

$$a_{\mu}(\vec{\kappa})\, a^{*}_{\mu'}(\vec{\kappa}') = \delta_{\mu\mu'}\,\delta_{\vec{\kappa}\vec{\kappa}'} \qquad \ldots(15)$$

The formula for the radiation probability per second then becomes

$$w_{nn'} = \frac{e^2 c}{\pi h}\int_{-\infty}^{\infty} dt\, e^{i c t \kappa_{nn'}} \iint d^3x_1\, d^3x_2\, G(t, \vec{r_1} - \vec{r_2})\, X \qquad \ldots(16)$$

$$X\, \psi^{*}_{n'}(r_1)\psi_{n'}(r_2)\left[\frac{E_n E_{n'}}{c^2} - \frac{\vec{P}(r_1)\cdot\vec{P}(r_2)}{n_o{}^2\; c^2}\right]\psi^{*}_{n}(r_2)\psi_n(r_1)$$

where $\kappa_{nn'} = (E_n - E_{n'})/\hbar c$ and the Green's function is given by

$$G(t,\bar{R}) = -\frac{1}{4\pi}\int \frac{d^3\kappa}{\kappa}\, e^{-i\kappa c t + i\vec{\kappa}\cdot\vec{R}}$$

$$= \left[c^2(t - i\epsilon)^2 - R^2\right]^{-1},\ \epsilon \to 0^{+} \qquad \ldots(17)$$

The imaginary component is to ensure proper deviation around poles on the real axis.

5. Applications of radiation formula

i) Rate of energy radiation by electron:

$$\frac{dE}{dt} = \sum_{n'} (E_{n'} - E_n)\, w_{nn'} \qquad \ldots(18)$$

Here we insert for $\Delta E = E_{n'} - E_n$,

$$\Delta E = \Delta \ell \left(\frac{dE}{d\ell}\right) + \Delta s \frac{dE}{ds} + \Delta k \frac{dE}{dk} \qquad \ldots(19)$$

and use Eq.(10) to obtain

$$\frac{dE}{dt} = -W^c\left[1 - \frac{55\sqrt{3}}{16}\;\frac{\hbar}{n_o c\, a}\;\left(\frac{E}{n_o c^2}\right)^2\right] \qquad \ldots(20)$$

Here W^c is the classical radiation rate, given in Lecture I. For the quantum correction to be important, the electron energy must be on the order of 10^3 Bev.

For this reason, one might at first sight suppose that quantum effects are unimportant in synchrotron operation. This is contradicted in the following section.

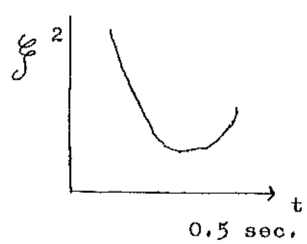
ii) Quantum effects on $\overline{\rho^2}$. Define for simplicity the adibatic invariant $\xi^2 = \overline{\rho^2}\ eH(a)\ \sqrt{1-q} = \hbar cs$. Then, as above,

$$\frac{d\xi^2}{dt} = hc \sum_{n'} (s' - s)\, w_{nn'}$$

$$= \frac{55}{48\sqrt{3}} \frac{e^2 h}{m_o a^2 (1-q)^{3/2}} \left(\frac{E}{m_o c^2}\right)^6 - \xi^2 \left(\frac{q}{1-q}\right) \frac{W}{E} \qquad \ldots(21)$$

The second term in Eq.(21) is the classical one, the first is quantum mechanical. Note the difference in sign: the classical term dampens the oscillations, the quantum term excites them. For a homogeneous field q=0, and only the first term occurs; it was obtained by Sokolov and Ternov in 1953. The classical term was found by Robinson and by Kolomensky and Lebedev. A German physicist named Gutbrod has recently been able to obtain the second term by quantum methods.

iii) Experimental observation of quantum term in ξ^2: by Sands (Cal Tech) and Korolev (Moscow) reported in the Nuovo Cimento 18, 1033 (1960). The latter used a 680-MeV synchrotron with an acceleration time of 0.6 sec. and took moving pictures of the beam with a camera speed of 500 frames/sec.



The resultant curve of ξ^2 as a function of time is shown. The first declining part of the curve is the classical damping; the second, rising part is the quantum excitation.

iv) Axial oscillations. The curve of $\overline{z^2}$ measured experimentally looks quite like that for ξ^2 except that the magnitudes are somewhat smaller. This is not in satisfactory agreement with theory, which predicts that in this case the quantum effect should be <u>much</u> smaller, so that no rising part of the curve would be observed. This discrepancy is not yet explained.

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Lecture III. QUASI-CLASSICAL INTERPRETATION

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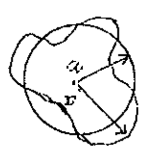
1. Introduction

In the last lecture, we obtained a formula (II,21) for the oscillations about the equilibrium orbit of a rotating electron. For electron energies $E \gtrsim 500$ MeV., these oscillations are substantially excited by the quantum emission of radiation and can be on the order of centimeters in magnitude. We thus have here a peculiar sort of macroscopic atom.

In the present lecture we give a quasi-classical interpretation of this situation. It is a simple means of taking the quantum effects into account and is of importance in the design of electron synchrotrons for energies of order 1 Bev. or more, where it was first introduced by Sands (Cal. Tech). It has the practical advantage that it can be employed in synchrotron design by persons who are not well versed in quantum mechanics. Here we do not consider technical details of machine design but rather general principles.

2. Classical radial equation

Let a be the equilibrium orbit radius for the electron, and define $\rho = r - a$ as the parameter to characterize the betatron oscillations about this orbit. The classical equation of motion is



$$\ddot{\rho} + \gamma \dot{\rho} + \omega_s^2 \rho = 0 \qquad \ldots(1)$$

where the dot indicates partial differentiation with respect to time. Here $\gamma = \left(\frac{q}{1-q}\right)\left(\frac{W}{E}\right)$, the classical radiation damping, where E = electron energy, W = radiation rate, q = field index in $H = br^{-q}$. The frequency of radial betatron oscillations is $\omega_s = \sqrt{1-q}\ \beta\,(c/a)$.

In this formula are no quantum effects, and the betatron oscillations follow the exponential decay given by putting $\hbar = 0$ in Eq.(II.21), when the first term of this equation vanishes. To obtain the specifically quantum effects, we should introduce an effective "quantum force" F^Q on the right hand side of Eq.(1).

3. The "quantum force"

We now try to devise an expression for this "quantum force". The equilibrium radius is determined by

$$\beta E = aeH(a) = eba^{1-q} \qquad \ldots(2)$$

Taking $\beta = 1$, we have

$$\frac{\Delta E}{E} = (1-q)\ (\Delta a/a) \qquad \ldots(3)$$

Radiation of a quantum with $\hbar\omega = \Delta E$ implies by Eq.(3) a corresponding instantaneous displacement of the center of betatron oscillation. If the quanta are radiated at random instants t_j, the "quantum force" is

$$F^Q = \sum_j \Delta a(t_j)\ \frac{d}{dt}\delta(t-t_j) \qquad \ldots(4)$$

where δ is the Dirac delta-function. (Note: if the electron itself received a sudden impulse, Eq.(4) would contain a δ-function; but since the center of oscillation received the sudden impulse instead, the derivative of the δ-function appears in Eq.(4). This question is discussed in the Classical Theory of Fields by Ivanenko and Sokolov).

Substituting for Δa from Eq.(3) and introducing a Fourier decomposition of the δ-function, we have

$$F^Q = \sum_j \frac{a\hbar\omega}{E(1-q)}\ \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ ik\ e^{ik(t-t_j)} \qquad \ldots(5)$$

Insertion of Eq.(5) on the right hand side of Eq.(1) leads, after manipulations standard for linear equations, to an expression for the fluctuation coordinate under quantum forces,

$$\rho^Q = \sum_j \frac{a\hbar\,\omega}{E(1-q)} \; \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{dk\ ik\ e^{ik(t-t_j)}}{\omega_s^{\,2}+ik\gamma-k^2}$$

$$= \sum_j \frac{a\hbar\,\omega}{E(1-q)} \begin{cases} e^{-\gamma/2(t-t_j)}\cos\,\omega_s(t-t_j), & t > t_j \\ 0 & , \; t < t_j \end{cases} \qquad \ldots(6)$$

where the second form applies under the approximation $\gamma << \omega_s$. The distinction in Eq.(6) between $t > t_j$ and $t < t_j$ is quite reasonable physically; for $t < t_j$ the jth quantum has not yet been emitted and so cannot affect ρ; but for $t > t_j$ the oscillations show the effect of previously emitted quanta.

4. Radial fluctuation

We can use the expression of Eq.(6) to compute the mean square radial fluctuation $\overline{\rho^2}$. For this purpose we assume the emission of different quanta to be statistically independent; viz.,

$$\overline{\cos\,\omega_s(t-t_j)\,\cos\,\omega_s(t-t_{j'})} \sim \delta_{jj'}, \qquad \ldots(7)$$

Then

$$\overline{\rho^2} = \rho_o^{\,2} e^{-\gamma t} + \int_0^t dt_j\,d\omega\,(\rho^Q)^2 \left\{\frac{W(\omega,t_j)}{\hbar\,\omega}\right\} \quad \ldots(8)$$

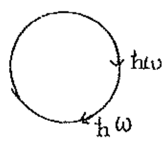
where $W(\omega,t_j)$ is the (classical) expression for the intensity of radiation per unit frequency, such that

$$\int_0^{\infty} W(\omega,t_j)\,d\omega = 2/3\,\frac{e^2c}{a^2}\,\left(E/m_oc^2\right)^4 \qquad \ldots(9)$$

The integral in Eq.(8) is not difficult to evaluate, and one obtains the same result as Eq.(II.21) from the pure quantum theory. There can be little argument about the correctness of this equation, either theoretically or experimentally. About the axial oscillations there may still be some question; but they are the same in principle and rather unimportant in practice, so we need not discuss them further.

5. Discussion of particles vs. waves

The present example provides a convenient starting point for a discussion of some general principles of quantum mechanics. We hope to show that the above relation of quantum and quasi-classical formulas is not just accidental.



In a betatron electrons follow classical laws while rotating in orbit; if radiation was absent, there would be no need for quantum mechanics. In fact, however, electrons emit quanta $\hbar\omega$ at unpredictable instants. Between such emissions, they follow classical orbits.

That is, the quantum excitation of betatron oscillations is associated with statistically independent, instantaneous radiation of photons. One might accordingly suggest that the electron is a classical entity with certain dimensions (Hofstadter's experiments), which continuously receives impulses due to radiation interaction and executes Brownian-type motion. Just as Brownian motion permits prediction only of <u>statistical</u> behavior of the particle subject to fluctuating impulses, so the electron motion -- especially in the examples above -- must be treated statistically because of sudden recoils from real, emitted photons.

6. Speculative comparison of macro-world and micro-world

i) The macro-world. Consider the non-relativistic equation of motion for a point electron in one dimension, examined by Dirac:

$$\dddot{x} - \gamma \ddot{x} = f(t) \qquad \ldots(10)$$

Here the classical damping coefficient is $\gamma = 2/3(e^2/m_oc^3) = 2/3(r_o/c)$, where r_o = classical electron radius. In the absence of a driving force, $f(t) = 0$, the general solution to Eq.(10) is

$$x = A + Bt + Ce^{\gamma t} \qquad \ldots(11)$$

There must be three constants, A, B, and C because Eq.(10) is of third order; but the term $Ce^{\gamma t}$ represents self-acceleration and is physically unreasonable. It must be eliminated by a suitable choice of initial conditions, for which we suggest the following, now becoming generally accepted:

$$x = x_o \ , \quad \dot{x} = v_o \quad \text{at } t = 0$$
$$\ddot{x} = 0 \qquad \text{at } t \longrightarrow \infty \qquad \ldots(12)$$

The last boundary condition means that all accelerations ultimately vanish in the infinite future, so that some sort of steady state is approached at least asymptotically.

Under the boundary conditions (12) the solution of Eq.(10) can be written (in terms of $\dot{x}$, since this is simpler than for x itself)

$$\dot{x}(t) = \int_{-\infty}^{t} f(t')\, dt' + \int_{t}^{\infty} f(t') e^{(t-t')/\gamma} dt' \qquad \ldots(13)$$

The first term of Eq.(13) would be present even for $\gamma = 0$ and can be interpreted as a retarded action of the driving force f(t) on the electron, since in the integral the time $t' < t$. The second term, however, is an advanced action, since $t' > t$ in the integral. It vanishes when $\gamma \longrightarrow 0$; but for $\gamma > 0$, the exponential factor allows contributions from f over a short, advanced time interval of order $\Delta t \approx \gamma$.

Thus one may say that the presence of the term in γ (radiation damping) causes the electron to be "smeared out" in time by an amount of order $\Delta t \approx \gamma$, so that it is slightly susceptible to advanced action. Another form of this interpretation is to note that an electromagnetic force f(t) would propagate a distance $c\Delta t \approx (2/3)r_o$ during this time; that is, we may assign the electron (originally assumed to be a point charge) "smearing out" in space that corresponds to its smearing out in time: namely, an effective radius of order r_o. This smearing out in space is due to the impulses the electron receives from the radiation.

Equation (10) and its solution (13) are completely classical; nevertheless, the language of the preceding paragraph very closely resembles that of the quasi-classical interpretation of quantum effects in synchrotron radiation, which we already know yields exactly the same results as strict quantum theory. We may thus hope to find a more direct connection between classical and quantum theory -- the macro-world and the micro-world -- an example of such possible connections being given in the next section.

ii) The micro-world. Let us write the driving force in Eq.(10) explicitly in electromagnetic terms:

$$\ddot{x} + \omega_o^2\, x - \gamma \dddot{x} = e\, E_x = -\frac{e}{c}\dot{A}_x \qquad \ldots(14)$$

We have added a normal frequency term $\omega_o^2\, x$, but this introduces no complications into this solution.

We decompose A_x into a Fourier series and assume second quantization, so that the components of this decomposition are non-commutative. The solution of Eq.(14) can be written

$$x = \int A_x(t')G_1(t-t')dt, \qquad \ldots(15a)$$

Likewise, for the momentum $p_x = m_o\dot{x} - \gamma m_o\ddot{x} + e/c\ A_x$ we have

$$p_x = \int A_x(t'')G_2(t-t'')dt'' \qquad \ldots(15b)$$

Here G_1 and G_2 are appropriate Green's functions.

Now since $A_x(t')$ and $A_x(t'')$ do not generally commute, one can evaluate the Heisenberg uncertainty relation for the electron,

$$p_x x - x p_x = i\hbar(1-2\gamma^2\omega_o^2) \qquad \ldots(16)$$

The second term on the right is an approximation for $\gamma^2\omega_o^2 \ll 1$; it remains always positive and < 1 for all γ, ω_o. In the limit when $\omega_o \longrightarrow 0$ this becomes just the usual uncertainty relation for a particle. But here we have obtained this result not from any postulates applied directly to the particle itself, but from the effects of quantum fluctuations induced on the particle by its interaction with the quantized electromagnetic field. This is the micro-world analogue of synchrotron oscillations in the macro-world.

This same procedure has been applied by Welton to vacuum fluctuation effects and yields the electrodynamic correction of $\alpha/2\pi$ to the electron magnetic moment, as well as the Lamb shift, etc. It might be of interest to consider the structure and magnetic moments of the proton and neutron from this point of view. One may also remark that extension of Eq.(13) to relativistic electrons in a synchrotron leads to a treatment of the clock paradox in relativity which has two features: the paradox can be entirely resolved, and it is not necessary to introduce gravitational fields.

SYNCHROTRON RADIATION

Lectures delivered at
The Australian National University
December 2-21, 1963
by
Professor A.A. SOKOLOV
Moscow State University

IV. RADIATION POLARIZATION PROPERTIES

1. Extraction of transverse components

For the radiation of real photons we must deal only with the transverse components of the electromagnetic field and must distinguish two different polarizations among these. Hence take

$$\varphi = 0\ , \quad \mathrm{div}\ \vec{A} = 0$$
$$H = \mathrm{curl}\ \vec{A}\ , \quad \vec{E} = -1/c\ \frac{\partial \vec{A}}{\partial t} \qquad \ldots(1)$$

These conditions assure the elimination of "scalar" and "longitudinal" photons.

To separate the transverse components into two circular polarizations take

$$\vec{H} = -is\vec{E} \qquad\qquad s = \pm 1 \qquad\qquad \ldots(2)$$

The quantity $s = \pm 1$ represents the component of photon spin along the direction of propagation.

The transverse, quantized field can then be written

$$\vec{A} = V^{-\frac{1}{2}} \sum_{\vec{k}} \frac{\sqrt{2\pi c\hbar}}{k} \left[\vec{b}(\vec{k},s)q(\vec{k},s)e^{-ickt + i\vec{k}\cdot\vec{r}} + \mathrm{c.c.}\right] \quad \ldots(3)$$

where q, q^+ are destruction and creation operators,

$$\left[q(\vec{k},s),\ q^+(\vec{k}',s')\right]_- = \delta_{\vec{k}\vec{k}'}\ \delta_{ss'} \qquad \ldots(4)$$

Since the photons satisfy Bose statistics we can define the number $N(\vec{k},s)$ of photons of momentum $\vec{k}$ and polarization s by

$$q^{+}(\vec{k},s)\,q(\vec{k},s) = N(\vec{k},s)$$

$$q(\vec{k},s)\,q^{+}(\vec{k},s) = N(\vec{k},s) + 1 \qquad \ldots(5)$$

In Eq.(3) the polarization vector is

$$\vec{b}(\vec{k},s) = \frac{1}{\sqrt{2}}\left(\vec{\beta}^{\,o} + is\vec{k}^{o} \times \beta^{o}\right) \qquad \ldots(6)$$

where $\vec{k}^{o} = \vec{k}/k$ is the unit vector in the direction of propagation, and $\vec{\beta}^{\,o}$ is a unit vector in any convenient transverse direction, $\vec{\beta}^{\,o}\cdot\vec{k}^{o} = 0$.

The energy in the radiation field is

$$\mathcal{H} = \frac{1}{8\pi}\int d^{3}x\,(E^{2}+H^{2}) = \sum_{\vec{k},s} \hbar ck N(\vec{k},s) \qquad \ldots(7)$$

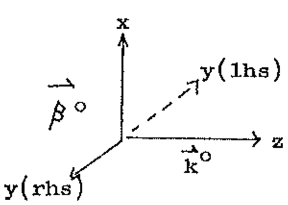
The total angular momentum is

$$\vec{S} = \frac{1}{4\pi c}\int d^{3}x\,(\vec{E}\times\vec{A}) = \sum_{\vec{k},s} \hbar\vec{k}^{o}s\, N(\vec{k},s) \qquad \ldots(8)$$

These are constants of the motion and should contain no time-dependent components.

2. Left- and right- circular polarization

The connection between the parameter $s = \pm 1$ and the circular polarization depends upon the choice of the coordinate system, either right- (rhs) or left- (lhs) handed. This is because the spin is an axial vector while the propagation vector is polar. Prevalent confusion on this point makes it worth expounding at some length. Throughout our discussion we adopt the following convention, suitable also for the Dirac equation: the direction of $\vec{k}^{o}$ is taken as the z-axis, that of $\vec{\beta}^{\,o}$ as the x-axis and that of $\vec{k}^{o}\times\vec{\beta}^{\,o}$ for the y-axis. This definition is valid for both (rhs) and (lhs) coordinates.



Consider now the time dependence in Eq.(3), where

$$b_{x}(t) \sim \mathrm{Re}\left\{e^{-ickt}\right\} = \cos\,(ckt) \qquad \ldots(9)$$

$$b_{y}(t) \sim \mathrm{Re}\left\{ise^{-ickt}\right\} = s\,\sin(ckt) = \cos(ckt - s\pi/2)$$

Therefore, in either (rhs) or (lhs) coordinates

$$s = +1 \quad \text{means} \quad y \longrightarrow x$$
$$s = -1 \quad \text{means} \quad x \longrightarrow y \qquad \ldots(10)$$

Equation (10) shows a physical meaning for s that is independent of the coordinate system.

It is more usual, however, to talk in terms of circular polarization, the direction of rotation of the spin as viewed by an observer being approached head-on by the photon (Note: this is the opposite of the usual convention in optics, where the light beam moves away from the viewer). This quantity is not independent of the choice of coordinate system, and we have

$$s^r = 1, \quad s^l = -1 \quad \text{for right circular polarization}$$
$$s^r = -1, \quad s^l = 1 \quad \text{for left circular polarization} \qquad \ldots(11)$$

where the superscripts r and l refer to the choice of coordinate system.

From now on we shall adopt a (rhs), so that $s = +1$ means right circular polarization, $s = -1$ means left circular polarization.

3. Linear polarization

It is also possible to express the transverse polarization in terms of two orthogonal linear polarizations. For this purpose we would decompose the vector potential as follows:

$$\vec{a} = \vec{a}_2 + \vec{a}_3$$
$$\vec{a}_2 = \vec{\beta}^o q_2, \quad \vec{a}_3 = (\vec{k}^o \times \beta^o) q_3 \qquad \ldots(12)$$
$$\vec{\beta}^o = (k^o \times j^o) / \sqrt{1-(k^o \cdot j^o)^2}$$

where $\vec{j}^o$ is any convenient unit vector in the system under consideration. The radiation intensity for each type of polarization is expressed in terms of the creation and destruction operators.

$$W_2 \sim q_2 q_2^+ , \quad W_3 \sim q_3 q_3^+ \qquad \ldots(13)$$

If there are no photons originally present, the creation and destruction operators satisfy

$$q_2 q_2^+ = q_3 q_3^+ = 1 , \quad q_2 q_3^+ = q_3 q_2^+ = 0 \qquad \ldots(14)$$

Transformation between the two systems of polarization (linear and circular) is made awkward by the existence of a phase difference between the two components, which cannot be given by the quantum mechanical formulae. To specify this phase difference is essentially a classical problem, for which we have recommended the following solution. The total energy radiated is of course the same in either polarization system:

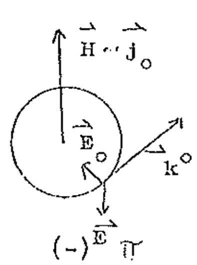
$$W = W_2 + W_3 = W_1 + W_{-1} \qquad \ldots(15)$$

where $W_{\pm 1}$ refer to circular polarization. Thus there are only three independent quantities among the four W_m ; and since the sums are constant, one expects that the differences will be related to the phases. In particular, it is found that

$$\sin \delta_{23} = \frac{W_{-1} - W_1}{2(W_2 W_3)^{\frac{1}{2}}} \qquad \ldots(16)$$

where δ_{23} is the phase difference between the two linearly polarized components. A corresponding formula exists for the phase difference between circular polarizations. The advantage of these formulas is that the phase difference can be evaluated directly from the (real) intensities W_m, for which quantum and classical expressions are immediately available.

4. Application to synchrotron radiation



The photon is emitted in a narrow cone almost tangential to the orbit; this is the direction $\vec{k}^o$. For $\vec{j}_o$ we take the direction of the applied magnetic field H, perpendicular to the electron orbit. The electric vectors of the two linear polarizations are

$\vec{E}_\sigma \sim \vec{k}_o \times \vec{j}_o$, pointing approximately toward the center of the orbit, and $\vec{E}_\pi \sim \vec{j}_o$ parallel to H.

Now write Schott's formula with polarization,

$$W_s(\nu,\theta) = \frac{e^2\beta^2 c\nu^2}{a^2}\left[s_2\beta J_\nu'(\nu\beta\sin\theta) - s_3\cot\theta\, J_\nu(\nu\beta\sin\theta)\right]^2 \qquad \ldots(17)$$

where ν is the harmonic number and θ the polar angle between $\vec{k}^o$ and $\vec{j}_o$.

For the σ-component, $s_2 = 1$, $s_3 = 0$,

$$W_\sigma(\nu,\theta) = \frac{e^2\beta^4 c\nu^2}{a^2} J_\nu'^2(\nu\beta\sin\theta) \qquad \ldots(18a)$$

and for the π-component, $s_2 = 0$, $s_3 = 1$,

$$W_\pi(\nu,\theta) = \frac{e^2\beta^2 c\nu^2}{a^2}\cot^2\theta J_\nu^2(\nu\beta\sin\theta) \qquad \ldots(18b)$$

The sum of Eqs.(18a) and (18b) is equivalent to Eq.(I,7) for the total radiation intensity.

For right or left circular polarization ($s = \pm 1$), we put $s_2 = \pm s_3 = 1/\sqrt{2}$ in Eq.(17). We can then compute the phase difference of the σ- and π-components according to Eq.(16); viz.,

$$\sin\delta_{\sigma\pi} = \frac{\cos\theta}{|\cos\theta|} \qquad \ldots(19)$$

This means that if $0 \leqq \theta < \pi/2$, $\sin\delta = 1$, and the radiation is left elliptically polarized; if $\pi/2 < \theta \leqq \pi$, $\sin\delta = -1$, and the radiation is right elliptically polarized. When $\theta = \pi/2$, the observer is in the plane of the orbit, and the radiation is linearly polarized with $W_\pi = 0$.

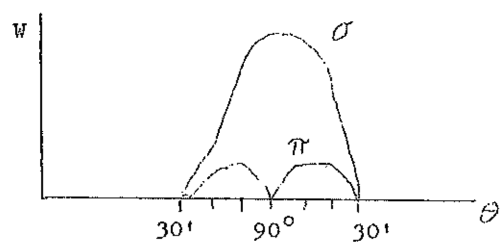
If we integrate over all angles and frequencies,

$$W_\sigma = (7/8)W, \quad W_\pi = (1/8)W \qquad \ldots(20)$$

so that most of the radiation is in fact σ-polarized (E_σ in the plane of the orbit).

5. Experimental check

Korolev first tested these polarization formulae experimentally on a 250 MeV. electron synchrotron. He checked the angular dependence of $W_\sigma(\nu,\theta)$ and $W_\pi(\nu,\theta)$ for a fixed ν or hence fixed wavelength of radiation, which was 4080Å for the spectrograph concerned. The spectrograph was fitted to distinguish σ- from π-polarization. The theoretical curves were computed by using the asymptotic form (I.11) for $J_\nu(X)$. The theoretical curve and experimental points were in very good agreement.



Later, this experiment was repeated independently at Cornell, with similar confirmation of the theory. Neither set of experiments was able to determine the relative phase of the σ and π components, i.e., the degree of elliptical polarization.

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Lecture V. ELECTRON POLARIZATION PROPERTIES

1. Dirac equations

The free-particle Hamiltonian due to Dirac is

$$H = c\,\rho_1(\vec{\sigma}\cdot\vec{p}) + \rho_3 m_0 c^2 \qquad \ldots(1)$$

Note: For consideration of spin effects, it is more convenient to use the Dirac matrices ρ_n, $\vec{\sigma}$ than the fully covariant γ_μ introduced by Pauli.

In the following, Greek indices run from 1 to 4, Latin indices from 1 to 3. With the help of the 4X4 identity matrix I, we can define

$$\rho_\mu = (\rho_n, I), \qquad \sigma_\mu = (\sigma_n, I) \qquad \ldots(2)$$

All of the 16 possible Dirac operators can be represented as simple products of one ρ and one σ ; viz.,

$$\begin{aligned}
&S: \quad \rho_3\sigma_4 = \rho_3 \\
&V: \quad \rho_1\vec{\sigma}, \ \rho_4\sigma_4 = I \\
&T: \quad \rho_3\vec{\sigma}, \ \rho_2\vec{\sigma} \qquad \ldots(3) \\
&A: \quad \rho_4\vec{\sigma} = \vec{\sigma}, \ \rho_1\sigma_4 = \rho_1 \\
&P: \quad \rho_2\sigma_4 = \rho_2
\end{aligned}$$

where S, V, T, A, P stand for scalar, vector, tensor, axial vector and pseudoscalar, respectively.

A difficulty with the Dirac Hamiltonian (1) is that it does not admit the spin $\vec{\sigma}$ as a constant of motion, but only the total angular momentum

$$\vec{J} = \vec{r}\times\vec{p} + \tfrac{1}{2}\hbar\vec{\sigma}$$
$$[H,\vec{J}] = 0 \qquad \ldots(4)$$

We wish to generalize the spin operator in the following way: The generalized operator should not contain any orbital angular momentum, should commute with H to the maximum possible extent, and should behave simply under Lorentz transformations and spatial rotation. We discuss three such generalizations:

i) generalization of the pseudovector A;

ii) generalization of the magnetic moment part of T;

iii) introduction of a unit spin vector with peculiar Lorentz transformation properties, such that it maintains 3-dimensionality (vanishing fourth component) in all Lorentz frames.

2. Generalization of A.

The pseudovector is $s_\mu = (\vec{\sigma}, \rho_1)$, in terms of which define

$$S_{\mu\nu} = \tfrac{1}{2}[\sigma_\mu P_\nu + P_\nu \sigma_\mu] \qquad \ldots(5)$$

where P_ν is the generalized momentum as defined in Eq.(II,2). Then the generalized operator is

$$S_\mu = S_{\mu 4} = \frac{1}{2c}\left[(H-e\varphi)\sigma_\mu + \sigma_\mu(H-e\varphi)\right] \qquad \ldots(6)$$

It is clear that $\mathcal{S}_\mu = \int d^3x\,\psi^+ S_\mu \psi = \int d^3x\,\psi^+ S_{\mu 4}\psi$ transforms as a Lorentz pseudovector.

For a free particle the components are

$$S = m_0 c\rho_3\vec{\sigma} + \rho_1\vec{p} \qquad \ldots(7)$$
$$S_4 = \vec{\sigma}\cdot\vec{p}$$

which are all constants of motion (commute with H); in the presence of interactions, only some of the corresponding components will commute with H.

3. Generalization of T

The tensor is $\alpha_{\mu\nu} = -\alpha_{\nu\mu} = \rho_3 \sigma, \rho_2 \sigma$; as before, define

$$F_{\mu\nu\lambda} = \tfrac{1}{2}(\alpha_{\mu\nu} P_\lambda + P_\lambda \alpha_{\mu\nu}) \qquad \ldots(8)$$

$$F_{\mu\nu} \equiv F_{\mu\nu 4} = ic \left[(H-e\varphi)\alpha_{\mu\nu} + \alpha_{\mu\nu}(H-e\varphi)\right]$$

Then $\mathcal{F}_{\mu\nu} = \int d^3x\, \psi^+ F_{\mu\nu} \psi = \int d^3x\, \psi^+ F_{\mu\nu 4} \psi$ behaves like an antisymmetric, second-rank tensor under Lorentz transformations.

As before, for a free particle the components of $F_{\mu\nu}$ commute with H; but in the presence of interactions, only some components are constants of the motion -- for example, the component of the magnetic moment parallel to an external magnetic field.

4. Unit spin vector

This concept is proving quite useful, although it exists only for the free-particle case. In terms of the S_μ above, define

$$\vec{u} = \frac{S_4 \vec{p}}{p^2} + \frac{\vec{p}\times(\vec{p}\times\vec{s})}{m_0 c p^2}$$

$$= \frac{\vec{p}\,(\vec{\sigma}.\vec{p})}{p^2} + \rho_3 \frac{\vec{\sigma}p^2 - \vec{p}(\vec{\sigma}.\vec{p})}{p^2} \qquad \ldots(9)$$

where the second form results from substituting the free-particle relations of Eq.(7). This vector is a peculiar mixture of the axial vector with the magnetic moment part of the tensor operator. The normalization of the unit vector $\vec{u}$ is the same as that of $\vec{\sigma}$; namely,

$$u_j^2 = 1, \quad \vec{u}.\vec{u} = 3 \qquad \ldots(10)$$

In order to see this more clearly, we may introduce a "natural" system of rectangular coordinates, in which the 3-axis is chosen along the direction of $\vec{p}$. Then

$$u_3 = \sigma_3 = \frac{\vec{\sigma}.\vec{p}}{p}$$

$$u_1,\ u_2 = \rho_3 \sigma_1,\ \rho_3 \sigma_2 \qquad \ldots(11)$$

5. Electron wave function

The Dirac equation for the motion of a free electron is

$$(i\hbar\, \partial/\partial\, t - H)\, \psi = 0 \qquad \ldots(12)$$

with H as in Eq.(1). The wave function ψ has four components: a two-fold distinction with respect to sign of energy and a two-fold distinction with respect to (arbitrary) spin direction.

We employ a supplementary condition

$$\left(\frac{\vec{\sigma}.\vec{p}}{p}\right) \psi = s\psi, \qquad s = \pm 1 \qquad \ldots(13)$$

This operator commutes with H and so is a constant of the motion; its eigenvalues $s = \pm 1$ denote apposite circular polarizations about the axis of the propagation vector, just as in the photon case. That is,

$$s^r = -s^l = 1 \quad \text{right helicity}$$

$$s^r = -s^l = -1 \quad \text{left helicity} \qquad \ldots(14)$$

Equation (14) is the exact analogue of Eq.(IV,11) except that the word "helicity" is used for electrons instead of "circular polarization". In analogy with Eq.(IV,9) we have

$$\rho_3 \sigma_2 \psi = is\, \rho_3 \sigma_1 \psi \qquad \ldots(15)$$

A Fourier expansion of ψ is written

$$\psi = V^{-\frac{1}{2}} \sum_s{}' \; c_s b_s \, e^{-ic\epsilon kt + i\vec{k}.\vec{r}} + c.c. \qquad \ldots(16)$$

where V is the volume of a large box, b_s is a four-component unit spinor, and C_s, C_s^+ are destruction and creation operators. Here $\epsilon = \pm 1$ specifies the sign of the energy; for the study of spin effects it is sufficient to take $\epsilon = +1$. We take the normalization of Eq.(16) appropriate to a single electron in an indefinite state of polarization,

$$\int \psi^+ \psi \, d^3x = C_1^+ C_1 + C_{-1}^+ C_{-1} = 1 \qquad \ldots(17)$$

6. Transformation properties of $\vec{u}$

The behavior of $\vec{u}$ under Lorentz transformation is most readily studied by use of Eqs.(16) and (17). Consider the quantities

$$U_1 = \int \psi^+ u_1 \psi \, d^3x = C_1^+ C_{-1} + C_{-1}^+ C_1$$

$$U_2 = \int \psi^+ u_2 \psi \, d^3x = i(C_{-1}^+ C_1 - C_1^+ C_{-1}) \qquad \ldots(18)$$

$$U_3 = \int \psi^+ u_3 \psi \, d^3x = C_1^+ C_1 - C_{-1}^+ C_{-1}$$

They satisfy the same normalization as the u_j, namely

$$U_j^2 = 1, \quad \vec{U} \cdot \vec{U} = 3 \qquad \ldots(19)$$

The quantities $\mathcal{S}_u$ following Eq.(6) can be expressed in terms of the U_j

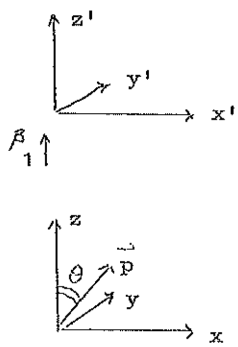
$$\mathcal{S}_3 = \omega U_3, \; \mathcal{S}_{1,2} = K \, U_{1,2}, \; \mathcal{S}_4 = k \qquad \ldots(20)$$

where $K = m_0 c/\hbar$, $k = p/\hbar$, $\omega = E/\hbar c$. Likewise, we can express the magnetic moment parts of $\mathcal{F}_{\mu\nu}$ following Eq.(8),

$$\mu_3 = K \, U_3, \; \mu_{1,2} = \omega \, U_{1,2} \qquad \ldots(21)$$

Now we see how it is that polarization properties for ultra-relativistic electrons must be considered in terms of $\vec{U}$ or by a peculiar mixture of $\vec{\mathscr{S}}$ and $\vec{\mu}$: when $\omega >> \kappa$, the longitudinal component $\mathscr{S}_3$ dominates over $\mathscr{S}_1$ and $\mathscr{S}_2$, while the dominant transverse components are μ_1, μ_2.

The Lorentz transformation of $\vec{U}$ can be obtained from the known pseudovector and tensor transformation laws for $\mathscr{S}_\mu$ and $\mathscr{F}_{\mu\nu}$, plus the relations in Eqs.(20) and (21); or the transformations may be determined directly from the definition in Eq.(9). This is particularly simple when one uses the "natural" coordinate system.



Suppose an unprimed system with the electron momentum vector $\vec{p}$ lying in the xz-plane and making an angle θ with the z-axis; let $\beta = k/\omega = pc/E$. Let the primed coordinate system move in the z-direction with a velocity $v_1 = c\,\beta_1$. In the unprimed system, the "natural" coordinates for $\vec{u}$ are such that U_3 is along the direction of $\vec{p}$; define U_2 to be in the y-direction, and U_1 to be perpendicular to both U_2 and U_3. In exactly the same way for the primed coordinate system, U_3' is along the direction of $\vec{p}\,'$, U_2' is in the y'-direction, and U_1' is perpendicular to U_2' and U_3'. Then it follows that

$$U_2' = U_2 \qquad \ldots(22a)$$

since these components remain perpendicular to the motion, while the transformation of U_3 and U_1 is determined by $\vec{p} \rightarrow \vec{p}\,'$; viz.,

$$U_3' = U_3 \cos\gamma + U_1 \sin\gamma$$

$$\ldots(22b)$$

$$U_1' = U_1 \cos\gamma - U_3 \sin\gamma$$

$$\cos\gamma = (\beta_1 - \beta\cos\theta) / \left[(\beta_1 - \beta\cos\theta)^2 + (1 - \beta_1^2)\,\beta^2 \sin^2\theta\right]^{\frac{1}{2}}$$

Here γ is the angle between $\vec{p}'$ and the z'-axis. Of course the transformation is unitary,

$$\vec{U}'.\vec{U}' = \vec{U}.\vec{U} = 3 \qquad \ldots(23)$$

7. Applications ; non-conservation of parity

By means of these formulae one can calculate spin effects in many phenomena: e.g., spin properties in elastic scattering at various energies, bremsstrahlung of polarized electrons, spin effects in synchrotron radiation. In the last mentioned case, the net effect is for synchrotron radiation to leave the electrons polarized in a direction opposed to the applied field $\vec{H}$. We have calculated that the electrons would be 90% polarized if they could run in a 1 Bev. synchrotron continuously for 1 hour. This has not yet proved possible in practice, but storage rings such as those developed at Frascati may make such observations possible. At present there are no experimental data on electron polarization in synchrotrons.

As a final application of spin considerations for the Dirac equation, I should like to consider failure of parity conservation, as introduced by Lee and Yang, and in particular the theory of the four-component neutrino as developed by myself and collaborators. If in the Dirac equation we set $m_o \equiv 0$, there are two possibilities:

i) The two-component neutrino (Lee-Yang, Landau, etc.), with the equation

$$\left\{E - c\,\vec{\sigma}'.\,\vec{p}\right\}\psi' = 0 \qquad \ldots(24)$$

where $\vec{\sigma}'$ is the two-component Pauli spin operator, and ψ' has only two components. Thus ψ' has one helicity for $E > 0$, the opposite helicity when $E < 0$; in practice, the neutrino has left helicity, the anti-neutrino has right helicity.

ii) The four-component neutrino (A.A. Sokolov et al.) has the equation

$$\left\{E - \rho_1\vec{\sigma}.\vec{p}c\right\}\psi = 0 \qquad \ldots(25)$$

on which we can impose the supplementary condition of Eq.(13); note that this condition is a Lorentz invariant for the special case $m_o = 0$. Now for <u>both</u> cases $E > 0$ and $E < 0$ we have two helicities. Thus there are four solutions :

$$\nu(\ell)\ ,\ \bar{\nu}(r)\quad \text{(as for 2-component case)} \qquad \ldots(26a)$$

$$\nu(r)\ ,\ \bar{\nu}(\ell)\quad \text{(equally good solution)} \qquad \ldots(26b)$$

With the recent experimental discovery that the electron and muon neutrinos are different, we find a case for both solutions in Eq.(20), taking (26a) to refer to ν_e and (26b) to refer to ν_μ. Then we have the consistent scheme

$$\begin{aligned} \text{leptons:}\quad & e^-, \mu^+\ ,\ \nu_e(\ell)\ ,\ \nu_\mu(r) \\ \text{anti-leptons:}\quad & e^+\ ,\ \mu^-,\ \bar{\nu}_e(r)\ ,\ \bar{\nu}_\mu(\ell) \end{aligned} \qquad \ldots(27)$$

Now simple conservation of leptons nicely avoids such unobserved reactions as $\mu \longrightarrow e + \gamma$. Of course one must understand that μ-decay is $\mu^- \longrightarrow e^- + \bar{\nu}_e(r) + \bar{\nu}_\mu(\ell)$ instead of $\mu^- \longrightarrow e^- + \nu + \bar{\nu}$, etc.

Finally, we should admit that this is not the only possible explanation of the known phenomena; but it is a point of view developed for some time by myself and my associates, and it seems worth presenting now because it fits in so well with the observed distinction between ν_e and ν_μ.